The Volterra Integral Equation of Second Kind

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THE THEORY OF THE VOLTERRA INTEGRAL EQUATION OF SECOND KIND

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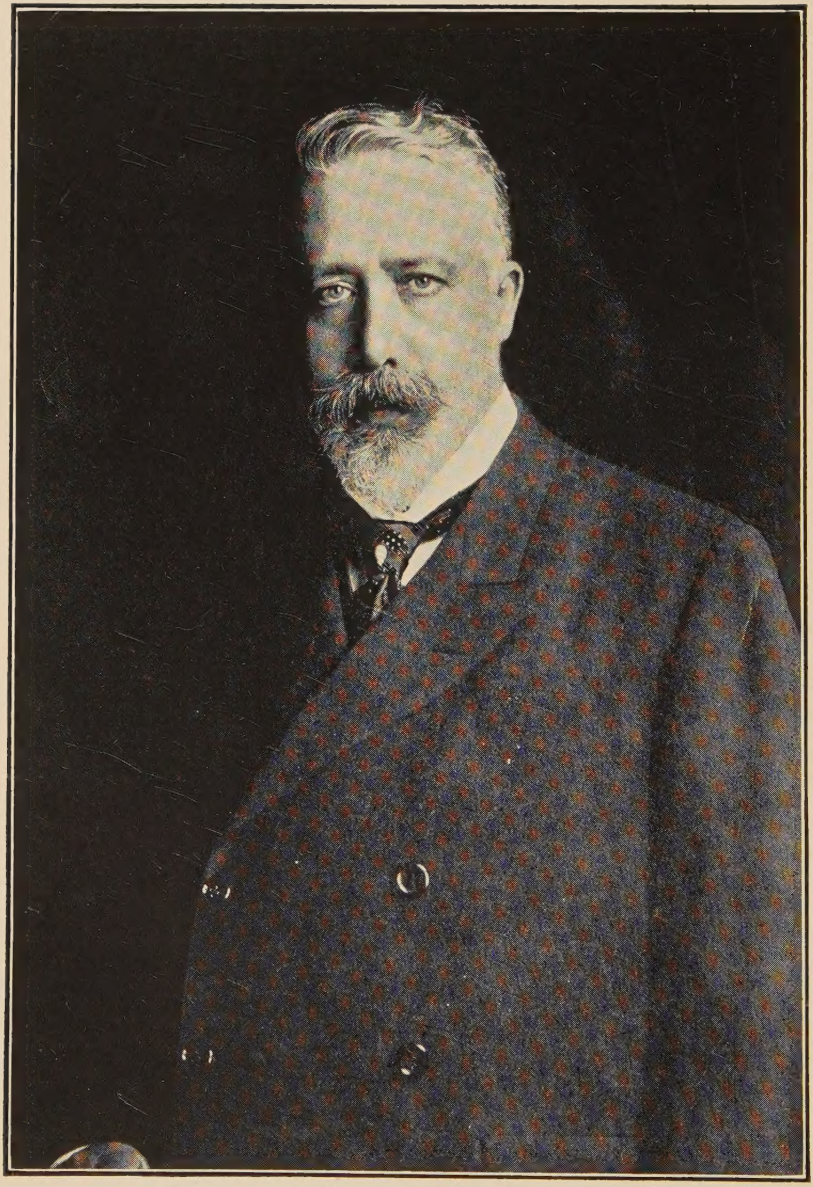


VITO VOLTERRA

The Theory of the Volterra Integral Equation of Second Kind

By Harold Thayer Davis, Associate Professor of Mathematics in Indiana University

INTRODUCTION

This paper is a treatment of the theory that underlies the solution of the Volterra integral equation of second kind, an equation so called because of the fundamental work of Vito Volterra in its development. By such an equation we shall mean the following:

$$u(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)\, dt, \tag{1}$$

where K(x,t), called the kernel, and f(x) are known functions defined over certain regions and where λ is a parameter the significance of which will appear as we proceed. The unknown function, the determination of which comprises the problem of the Volterra equation, is u(x).

Perhaps the first example of the use of such an equation is found in the work of S. Poisson where it appears in connection with his theory of induced magnetism. The ideas which Poisson developed there have historical significance in our subject and may be briefly stated as follows:*

Denoting the components of the magnetic force by H_x,H_y, and H_z, and those of the magnetization by h_x, h_y, and h_z, Poisson assumed a relationship between them defined by three equations of the type,

$$h_x(t) = \int_0^t K(t-s)\, H_x(s)\, ds \quad , \tag{2}$$

where t is the time variable and K(z) is a known function.

Letting V be the potential due to the magnetism external to the body, the magnetization of which is studied, and U the potential due to the induced magnetism, Poisson further assumed that the components of the two vectors satisfy three equations of the type,

$$H_x = -\partial V/\partial x - \partial U/\partial x + 4\pi k\, h_x/3 \quad ,$$

*Mémoire sur la théorie du magnétisme en mouvement. Mémoires de l'académie royale des sciences de l'institut de France, vol. 6 (1823), pp. 441-570; also Oeuvres, vol. 3, pp. 41-72. See also H. Bateman, reference 474, Messenger of Mathematics, vol. 39 (1909), pp. 173-178 and Clerk Maxwell, Electricity and Magnetism, vol. 2, (3d edition), pp. 50-58.

where k represents the ratio of the volume of magnetic elements to the total volume of the body, and is connected with the magnetic permeability, μ, by the equation $k = (\mu - 1)/(\mu + 2)$.*

From the fact that the components of the vector $\mathbf{h} = (h_x, h_y, h_z)$ satisfy the two vector equations, $\nabla \cdot \mathbf{h} = 0$, $\nabla \times \mathbf{h} = 0$, where ∇ is the vector operator, $\nabla = \mathbf{i}\partial/\partial x + \mathbf{j}\partial/\partial y + \mathbf{k}\partial/\partial z$, we infer that there exists a potential function, ϕ, such that

$$h_x = \partial\phi/\partial x\,, \qquad h_y = \partial\phi/\partial y\,, \qquad h_z = \partial\phi/\partial z\,.$$

In terms of this function we may then write the potential U as the integral,

$$U = \iint [(\partial\phi/\partial n) / r]\, dS\,,$$

where the integration is over the surface of the magnetized body, $\partial\phi/\partial n$ is the normal derivative of ϕ, and r is the distance from a surface element to the point at which the value of U is being computed.

Substituting the values of h_x and H_x in (2) we obtain for the determination of $\partial\phi/\partial x$ an equation of the form,

$$\partial\phi(t)/\partial x = -\int_0^t K(t-s) \left\{ \partial V(s)/\partial x + \partial U(s)/\partial x \right\} ds$$

$$+ (4\pi k/3) \int_0^t \left\{ K(t-s)\ \partial\phi(s)/\partial x \right\} ds\,.$$

This is clearly an integral equation of type (1) except for the simplification of the kernel which is a function of t—s. Such equations are often referred to as integral equations of Poisson type, although they are also called equations of the closed cycle because of their intimate connection with fundamental problems in hysteresis and hereditary mechanics.†

Poisson gave a solution of this equation by a method of successive approximation, but did not consider the convergence of his solution. This deficiency was later remedied by J. Liouville.‡

The material in the present study supplements that of a former paper entitled: "A Survey of Methods for the Inversion of Integrals of Volterra Type." The treatment has been divided into two parts, the first of which includes, in addition to a summary of the methods used by Volterra, other more specialized devices employed in the solution of equation (1). Part II has been devoted to a systematic

*For soft iron k is too close to unity to make probable the theory of Poisson in this particular.

†See the author's study: The Present Status of Integral Equations, Indiana University Study No. 70, pp. 26-27.

‡Reference 53.

development of the case of the singular integral equation which presents problems of unusual difficulty. In this the fundamental memoirs of G. C. Evans and an important extension by C. E. Love have been surveyed. The case where the kernel is developable into a series of the form,

$$K(x\,,\,t) = a_0(t) + a_1(t)\,(t—x)/1! + \ .\ .\ . + a_n(t)\,(t—x)^n/n\,! + \ .\ .\ .\ .\ ,$$

has been only casually treated because of the fuller discussion already given in the former study. The theorems which there apply to the inversion problem of the Volterra equations of first kind are easily carried over to the equation considered in this study.

No attempt has been made to examine the case of non-linear Volterra equations, Volterra equations of two or more variables, integro-differential equations, or systems of Volterra equations. Because of easily accessible treatments of the calculus of functions of composition this subject has been introduced only to the extent of its usefulness in its immediate application to Volterra equations.

The author reserves for a later project the systematic development of the relationship which exists between linear functional equations and algebraic systems of equations. This powerful method of investigating transcendental problems, which originally led Volterra and later Fredholm to their results, deserves a more extended treatment than can be given in the scope of the present study.*

References are given to correspond to the numbers in the bibliography of the author's paper on: "The Present Status of Integral Equations", and are repeated at the end of this study.

*See reference 19, p. 40, and reference 251, Atti di Torino, vol. 31, p. 311 et seq. See also R. D. Carmichael: Algebraic Guides to Transcendental Problems. Bulletin of the Amer. Math. Soc., vol. 28 (1922), pp. 179-210.

PART I.

GENERAL METHODS OF SOLUTION.

1. The Method of Successive Substitutions. We shall begin by assuming that in the equation to be studied,

$$u(x) = f(x) + \lambda \int_a^x K(x\,,\, t)\, u(t)\, dt, \tag{1}$$

the function f(x) is continuous in an interval (a, b) and K(x, t) is continuous in both variables over the triangle $a \leqq t \leqq x \leqq b$. Under these conditions we can apply the method of successive substitutions to equation (1).

Let us define an operator $S(\phi)$ by means of the integral:

$$S(\phi) = \int_a^x K(x\,,\, t)\, \phi(t)\, dt.$$

It immediately follows that the operator has the following properties:

(*a*) If ϕ is continuous over (a,b), $S(\phi)$ must be continuous also;

(*b*) $S(c\phi) = c\, S(\phi)$, where c is a constant;

(*c*) $S\left(\sum_{i=1}^{m} c_i\, \phi_i\right) = \sum_{i=1}^{m} c_i\, S(\phi_i)$, where the c_i are constants.

The following fundamental result is easily obtained:

Theorem 1. If in equation (1), f(x) is continuous over an interval (a,b) and K(x,t) is continuous over the triangle $a \leqq t \leqq x \leqq b$ then there exists one and only one continuous solution u(x), which is given by the formula,

$$u(x) = S^0(f) + \lambda S(f) + \lambda^2 S^2(f) + \lambda^3 S^3(f) + \ldots\ldots\ldots, \tag{2}$$

where we employ the abbreviations,

$$S^0(f) = f(x)\,, \quad S^n(f) = S[S^{n-1}(f)]\,, \quad n = 1, 2, 3, \ldots\ldots.*$$

To prove this theorem write (1) symbolically as,

$$u = f + \lambda\, S(u)\,.$$

Then making one substitution of u in the right-hand member and using property (c) we have,

$$u = f + \lambda S[f + \lambda S(u)] = f + \lambda S(f) + \lambda^2 S^2(u)\,.$$

*This theorem can be extended by admitting a finite set of finite discontinuities for f(x) and by making the hypothesis that the discontinuities of K(x,t) are regularly distributed. The solution will then be continuous except for a finite set of values of x and will remain bounded. For a discussion of this case see Part II.

If this process of substitution be continued indefinitely there results,

$$u = f + \lambda S(f) + \lambda^2 S^2(f) + \lambda^3 S^3(f) + \ldots\ldots \quad (3).$$

It remains to be proved, first, that this series is uniformly convergent; second, that it is a solution of equation (1); and, third, that the solution thus defined is unique.

The uniform convergence is easily established by observing the inequality,

$$| S(f) | \leqq FK | x - a | \leqq FK (b - a) ,$$

where F is the maximum value of f(x) in the interval and K the maximum value of K(x,t) in its triangle of definition.

We then have,

$$| S^2(f)| \leqq \int_a^x | K(x , t) | FK | t - a | |dt| \leqq FK^2 (b-a)^2/2 ! \quad ,$$

and in a similar way,

$$| S^n(f) | \leqq FK^n (b-a)^n/n !$$

Since the last expression is the nth term of the expansion of $Fe^{K(b-a)}$, series (2) is uniformly convergent for all values of x in (a,b) and represents an entire function in λ.

To prove that the series is a solution of the original equation we show that its substitution in $I(u) \equiv u(x) - \lambda \int_a^x K(x,t)u(t)dt$ reduces this expression to f(x).

Since the series converges uniformly we may write,

$$I(u) \equiv \sum_{n=0}^{\infty} \lambda^n S^n(f) - \lambda \int_a^x K(x , t) \left[\sum_{n=0}^{\infty} \lambda^n S^n(f) \right]_{x=t} dt ,$$

$$\equiv \sum_{n=0}^{\infty} \lambda^n S^n(f) - \sum_{n=0}^{\infty} \lambda^{n+1} \int_a^x K(x , t) \left[S^n(f) \right]_{x=t} dt ,$$

$$\equiv \sum_{n=0}^{\infty} \lambda^n S^n(f) - \sum_{n=0}^{\infty} \lambda^{n+1} S^{n+1}(f) ,$$

$$\equiv S^o(f) + \sum_{n=1}^{\infty} \lambda^n S^n(f) - \sum_{n=1}^{\infty} \lambda^n S^n(f) \equiv f(x) .$$

That the solution thus obtained is unique is easily established. If this were not the case then the difference between two solutions, which we may designate by v(x), would be a solution of the homogeneous equation,

$$v(x) = \lambda \int_a^x K(x , t) \, v(t) \, dt \quad . \quad (4)$$

Writing equation (4) in the symbolic form, $v(x) = \lambda S(v)$, we see by continuous substitution that $v(x)$ must also satisfy the following sequence of equations:

$$v(x) = \lambda^2 S^2(v)\ ,\quad v(x) = \lambda^3 S^3(v)\ ,\ \ldots\ldots\ ,\quad v(x) = \lambda^n S^n(v)\ .$$

But we have already proved that $\lim_{n=\infty} S^n(f) = 0$ for all bounded functions. Since $v(x)$ is such a function it must consequently be identically zero.

Example 1. As an example consider the solution of the following equation:

$$u(x) = 1 + \lambda \int_0^x (t - x)\, u(t)\, dt\ .$$

Computing the values of the operators we have,

$$S(1) = \int_0^x (t-x)\, dt = -x^2/2\,!\ ,\quad S^2(1) = \int_0^x (t-x)\,(-t^2/2\,!)\, dt = x^4/4!\ ,$$

$$S^3(1) = -x^6/6\,!\ ,\quad \ldots\ldots\ ,\quad S^n(1) = (-1)^n\, x^{2n}/(2n)\,!\ ,\ \ldots\ldots\ .$$

The solution is thus found to be,

$$u(x) = 1 - \lambda x^2/2! + \lambda^2 x^4/4\,! - \lambda^3 x^6/6\,! + \ldots\ldots\ ,$$
$$= \cos \sqrt{\lambda}\, x\ .$$

Example 2. (Giuseppe Usai, reference 472).

Solve the equation,

$$u(x) = x + \int_0^x (x^a - t^a)\, u(t)\, dt\ .$$

Computing the values of the operators we get,

$$S(x) = \int_0^x (x^a - t^a) t\, dt = a\, x^{a+2}/2(2+a)\ ,\quad S^2(x) = a^2\, x^{2a+3}/2(2+a)(3+a)(3+2a)$$

$$\ldots\ldots\ ,\ S^n(x) = a^n\, x^{n(a+1)+1}/2(2+a)\,(3+a)\,(3+2a)\ldots[(n+1) + (n-1)a]\,[(n+1) + na]\ .$$

Since, moreover, we have,

$$S^{n+1}(x)/S^n(x) = a\, x^{a+1}/[(n+2) + na]\,[(n+2) + (n+1)a]\ ,$$

which approaches zero with increasing n, the convergence of the series $u(x) = \sum_{n=0}^{\infty} S^n(x)$ is assured. For the special case $a = 1$, we get

$u(x) = \sinh x$.

Similarly the solution of the equation,

$$u(x) = x - \int_0^x (x^a - t^a)\, u(t)\, dt\,,$$

is $u(x) = \sum_{n=0}^{\infty} (-1)^n S^n(x)$. When $a = 1$, this reduces to $u(x) = \sin x$.

Example 3. (Usai)

Solve the equation,

$$u(x) = 1 + \int_0^x (x^a - t^a)\, u(t)\, dt\,.$$

The solution is easily proved to be,

$$u(x) = 1 + \sum_{n=1}^{\infty} a^n x^{n(a+1)}/(1+a)\,(2+a)\,(2+2a)\,(3+2a)\ldots[n+(n-1)a]\,[n+na].$$

If the sign before the integral is negative the solution is obtained by inserting $(-1)^n$ into the nth term. For $a = 1$, the solution reduces to $u(x) = \cosh x$ and when the sign before the integral is negative to $u(x) = \cos x$.

Example 4. Solve the equation,

$$u(x) = x^a + a^a \int_0^x (x - t)\, u(t)\, dt\,.$$

By means of successive substitutions we find the solution to be,

$$u(x) = x^a + \sum_{n=1}^{\infty} a^{na}\, x^{a+2n}/(a+1)\,(a+2)\ldots(a+2n)\,.$$

If the sign of the kernel is reversed, then the solution will be obtained by inserting $(-1)^n$ into the nth term.

For $a = 2$, the solution reduces to $u(x) = \sinh^2 x$; for $a = 2$ and the sign of the kernel reversed, $u(x) = \sin^2 x$.

Theorem 1 may be called the fundamental existence theorem for the Volterra equation of second kind. It is to be noticed, however, that the region of definition is limited to the interval $a \leqq x \leqq b$ and to the triangle $a \leqq t \leqq x \leqq b$. G. C. Evans* has employed the following device to extend the solution known for the interval (ab) into the interval beyond b.

Thus let us write equation (1) in the form,

$$u(x) = g(x) + \int_b^x K(x, t)\, u(t)\, dt\,,$$

*Reference 436, p. 662.

$$g(x) = f(x) + \int_a^b K(x, t)\, u(t)\, dt\,.$$

Since by hypothesis u(x) is known in the interval (ab), the function g(x) can be determined. The extended equation is then amenable to treatment by the usual methods.*

2. The Method of Resolvent Kernels. A second method for solving equation (1) was devised by Volterra. This method especially recommends itself because, when once the resolvent function associated with the kernel has been found, the solution of the equation for any function f(x) is at once reducible to a quadrature. This is not the case in the last method, which requires new computations for every new function f(x).

We first consider the sequence of functions:

$$K_1(x, y) = K(x, y)\,, \quad K_2(x, y) = \int_y^x K(x, t)\, K_1(t, y)\, dt\,,$$

$$\ldots\ldots\ldots\,, \quad K_n(x, y) = \int_y^x K(x, t)\, K_{n-1}(t, y)\, dt\,,$$

which may be referred to as the *successive iterated kernels of* $K(x,y)$.

Volterra has called the operation,

$$\psi = \int_x^y f(x, t)\, g(t, y)\, dt\,,$$

the composition of two functions if f(x,t) and g(t,y) are integrable and has invented for it the symbolic notation,

$$\psi = \overset{*}{f}\,\overset{*}{g}\,.$$

Altho an extensive theory of functions of composition has been developed, the particular property of them with which we are concerned in the present problem is the following:

Any integrable function of two variables, f(x,y), satisfies the symbolic index law,

$$\overset{*}{f}^{m+n} = \overset{*}{f}^{m}\ \overset{*}{f}^{n}\,,$$

where m and n are both integers.†

*This device has also been used by L. Orlando for the investigation of solutions in the neighborhood of the point b by means of the method of successive approximation. His application referred especially to kernels of the form: $K(x,t) = p_1(x)\, q_1(t) + p_2(x)\, q_2(t) + \ldots + p_n(x)\, q_n(t)$. See reference 471.

†Consult references 191, 490, 495.

To prove this identity, consider the iterated kernel,

$$K_3(x, y) = \int_y^x K(x, t)\, K_2(t, y)\, dt ,$$

$$= \int_y^x K(x, t) \int_y^t K(t, s)\, K(s, y)\, ds\, dt .$$

Applying to this iterated integral the Dirichlet formula for the interchange of the order of integration,* we shall have,

$$K_3(x, y) = \int_y^x K(s, y) \int_s^x K(x, t)\, K(t, s)\, dt\, ds ,$$

$$= \int_y^x K_2(x, s)\, K(s, y)\, ds .$$

It thus follows that we can write,

$$\int_y^x K(x, t)\, K_2(t, y)\, dt = \int_y^x K_2(x, t)\, K(t, y)\, dt ,$$

and by an identical argument that,

$$K_n(x, y) = \int_y^y K(x, t)\, K_{n-1}(t, y)\, dt = \int_y^x K_{n-1}(x, t)\, K(t, y)\, dt. \qquad (5)$$

Functions which possess this property, namely,

$$\overset{\bullet}{f}\,\overset{\bullet}{g} = \overset{\bullet}{g}\,\overset{\bullet}{f} ,$$

are called *permutable functions.*

We can now establish by induction the following important theorem which generalizes (5) and from which we derive the statement of the index law of functions of composition.

Theorem 2. If $K(x,t)$ is integrable in the triangle $a \leqq t \leqq x \leqq b$ then we have the following identity,

$$K_h(x, y) = \int_y^x K_i(x, t)\, K_{h-i}(t, y)\, dt , \quad i = 1, 2, \ldots\ldots, h - i ,$$

where $K_h(x,y)$ is the h-th iterated kernel.

We have already shown that this is true for $i = 1$. We now assume that it is true for $i = k$ and establish it for $i = k + 1$. The proof then follows by induction. Let us then assume that we have,

$$K_h(x, y) = \int_y^x K_k(x, t) \int_y^t K(t, s)\, K_{h-k-1}(s, y)\, ds\, dt .$$

*See footnote to page 24, A Survey of Methods for the Inversion of Integrals of Volterra Type.

By Dirichlet's formula for the interchange of the order of integration we have,

$$K_h(x, y) = \int_y^x K_{h-k-1}(s, y) \int_s^x K_k(x, t)\, K(t, s)\, dt\, ds ,$$

$$= \int_y^x K_{k+1}(x, s)\, K_{h-k-1}(s, y)\, ds ,$$

which establishes the theorem for subscript $k+1$.

Translating this result into the notation of functions of composition we have,

$$\overset{*}{K}{}^h = \overset{*}{K}{}^i \overset{*}{K}{}^{h-i} ,$$

which with proper identification of letters is seen to be equivalent to the index law stated above.

If we now return to solution (2) of the preceding section we see that we can simplify the expression of the operators. We thus have,

$$S(f) = \int_a^x K(x, t)\, f(t)\, dt ,$$

$$S^2(f) = \int_a^x K(x, t) \int_a^t K(t, s)\, f(s)\, ds\, dt ,$$

$$= \int_a^x f(s) \int_s^x K(x, t)\, K(t, s)\, dt\, ds ,$$

$$= \int_a^x K_2(x, t)\, f(t)\, dt .$$

Similarly we get,

$$S^3(f) = \int_a^x K_3(x, t)\, f(t)\, dt ,$$

. . . . . .

$$S^n(f) = \int_a^x K_n(x, t)\, f(t)\, dt .$$

Substituting these values in equation (2) we obtain Volterra's solution, which may be stated thus:

Theorem 3. The solution of equation (1) under the restrictions imposed in theorem 1, may be written in the form,

$$u(x) = f(x) + \lambda \int_a^x k(x, t)\, f(t)\, dt ,$$

where we use the abbreviation, $k(x,t) = K(x,t) + \lambda K_2(x,t) + \lambda^2 K_3(x,t) +$ . . . . . .

This new function, k(x,t), is called the *resolvent* or *solving* kernel.

Example 1. As an example consider the equation,

$$u(x) = f(x) + \lambda \int_0^x (t - x)\, u(t)\, dt ,$$

which includes the example of the preceding section as a special case. The iterated kernels are found to be,

$$K_1(x, y) = y - x , \quad K_2(x, y) = \int_y^x (t - x)\,(y - t)\, dt = (x - y)^3/3 ! ,$$

$$K_3(x, y) = \frac{1}{3!} \int_y^x (t - x)\,(t - y)^3\, dt = -(x - y)^5/5! , \quad . . . . . ,$$

$$K_n(x, y) = (-1)^n\, (x - y)^{2n-1}/(2n - 1) ! .$$

Hence we get for the resolvent kernel the function,

$$k(x, y) = -[(x - y) - \lambda(x - y)^3/3 ! + \lambda^2 (x - y)^5/5 ! - \lambda^3 (x - y)^7/7 ! + . . .]$$

$$= -\left\{ \sin \sqrt{\lambda}\,(x - y) \right\} / \sqrt{\lambda} ,$$

from which we obtain the solution,

$$u(x) = f(x) + \sqrt{\lambda} \int_0^x \sin \sqrt{\lambda}\,(t - x)\, f(t)\, dt .$$

Example 2. Consider the equation,

$$u(x) = f(x) + \lambda \int_0^x (x^a - t^a)\, u(t)\, dt .$$

We have seen in the examples of the preceding section how this equation is solved for special determinations of f(x). The general procedure is as follows:

$$K(x, y) = x^a - y^a ; \quad K_2(x, y) = \int_y^x (x^a - t^a)\,(t^a - y^a)\, dt =$$

$$\left\{ x^{2a+1} - a\,(2a+1)\, x^a y^{a+1} + a\,(2a+1)\, x^{a+1} y^a - y^{2a+1} \right\} / (a+1)\,(2a+1) ;$$

$$K_3(x, y) = a^2 \left\{ x^{3a+2} - [(2a+1)\,(3a+2)\, x^{2a+2} y^a / (a+2)] + [(2a+1)\,(3a+2)\right.$$

$$\left. x^a y^{2a+2} / (a+2)] - 2(3a+2)\, x^{a+2} y^{2a+1} + 2\,(3a+2)\, x^{2a+1}\, y^{a+1} - y^{3a+2} \right\} /$$

$$(a+1)\,(2a+1)\,(2a+2)\,(3a+2) .$$

$$K_4(x, y) = a^3\{x^{4a+3} - [(2a+1)\ (3a+2)\ (4a+3)\ x^{3a+3}\ y^a/(a+2)\ (2a+3)]$$
$$+ 3(4a+3)\ x^{3a+2}y^{a+1} - [3(3a+2)\ (4a+3)\ x^{2a+2}y^{2a+1}/(a+2)] + [3(3a+2)$$
$$(4a+3)\ x^{2a+1}y^{2a+2}/(a+2)] - 3(4a+3)\ x^{a+1}y^{3a+2} + [(2a+1)\ (3a+2)\ (4a+3)$$
$$x^a\ y^{3a+3}/(a+2)\ (2a+3)] - y^{4a+3}\}/(a+1)\ (2a+1)\ (2a+2)\ (3a+2)\ (3a+3)\ (4a+3).$$

. . . . . .

It is important to observe from the results of theorem 2 that the kernel and its resolvent may be regarded as mutual solutions of the following integral equations:

$$k(x, y) = K(x, y) + \lambda\int_y^x K(x, t)\ k(t, y)\ dt\ , \qquad (6)$$
$$= K(x, y) + \lambda\int_y^x k(x, t)\ K(t, y)\ dt\ ,$$

which may be written in the notation of the calculus of functions of composition as,

$$k = K + \lambda\ \overset{*}{K}\ \overset{*}{k}\ ,$$
$$= (\overset{*}{1^o} + \lambda\overset{*}{k})\ \overset{*}{K}\ ,$$

or in the second case as,

$$k = \overset{*}{K}\ (\overset{*}{1^o} + \lambda\overset{*}{k})\ .$$

The proof is easily supplied from the relations,

$$K_n(x, y) = \int_y^x K(x, t)\ K_{n-1}(t, y)\ dt = \int_y^x K_{n-1}(x, t)\ K(t, y)\ dt\ .$$

For the first equation in (6) we use the first of these identities as follows:

$$k(x, y) = K(x, y) + \sum_{n=2}^{\infty} \lambda^{n-1}K_n(x, y)\ ,$$
$$= K(x, y) + \sum_{n=2}^{\infty} \lambda^{n-1}\int_y^x K(x, t)\ K_{n-1}(t, y)\ dt\ ,$$
$$= K(x, y) + \lambda\int_y^x K(x, t)\ \sum_{n=2}^{\infty} \lambda^{n-2}K_{n-1}(t, y)\ dt\ ,$$
$$= K(x, y) + \lambda\int_y^x K(x, t)\ k(t, y)\ dt\ .$$

If the kernel has the simpler form, $K(x,y) = K(x—y)$, then equation (6) may be reduced. Since all the iterated kernels are functions of x—y, the resolvent is of the form k(x—y) and thus we have,

$$k(x—y) = K(x—y) + \lambda\int_y^x K(x—t)\, k(t—y)\, dt\ .$$

Making the substitutions $x—y = z$ and $t—y = s$ we get for the first equation in (6),

$$k(z) = K(z) + \lambda\int_o^z K(z—s)\, k(s)\, ds \quad , \tag{7}$$

and for the second equation,

$$k(z) = K(z) + \lambda\int_o^z k(z—s)\, K(s)\, ds \quad . \tag{8}$$

G. C. Evans has investigated the problem of finding the differential equation of which k(x,y) is a solution, when the differential equation satisfied by K(x,y) is known.*

Let us suppose that K(x,y) is a solution of the equation,

$$D_x \rightarrow K(x, y) = 0 \quad ,$$

where D_x is the operator defined by the series,

$$D_x = a_o(x) + a_1(x)\partial/\partial x + a_2(x)\partial^2/\partial x^2 + \ldots + a_n(x)\, \partial^n/\partial x^n \quad ,$$

and the arrow is the symbol of operation.

Applying this operator to equation (6) we get,

$$D_x \rightarrow k(x, y) = D_x \rightarrow K(x, y) + \lambda\int_y^x D_x \rightarrow K(x, t)\, k(t, y)\, dt$$

$$+ \lambda D_1 \rightarrow K(x, x)\, k(x, y) + \lambda D_2 \rightarrow K'_x(x, x)\, k(x, y) + \lambda D_2 \rightarrow K''_x(x, x)\, k(x, y)$$
$$+ \ldots\ldots\ldots \quad ,$$

where we use the abbreviation,

$$D_i = a_i + a_{i+1}\partial/\partial x + a_{i+2}\partial^2/\partial x^2 + \ldots + a_n\partial^{n-i}/\partial x^{n-i}.$$

Since by hypothesis $D_x \rightarrow K(x,y) = 0$, we find the equation satisfied by k(x,y) to be:

$$D_x \rightarrow k(x, y) — \lambda[D_1 \rightarrow K(x, x)\, k(x, y) + D_2 \rightarrow K'_x(x, x)\, k(x, y) +$$
$$D_3 \rightarrow K''_x(x, x)\, k(x, y) + \ldots + D_n \rightarrow K_x^{(n-1)}(x, x)\, k(x, y)] = 0 \quad . \tag{9}$$

For the case where the kernel is a function of $z = x—y$, the last equation reduces to,

$$D_z \rightarrow k(z) — \lambda[K(0)D_1 \rightarrow k(z) + K'(0)D_2 \rightarrow k(z) +$$
$$K''(0)D_3 \rightarrow k(z) + \ldots + K^{(n-1)}(0)D_n \rightarrow k(z)] = 0 \quad , \tag{10}$$

or in terms of the coefficients of the operator,

*Reference 380, p. 456 et seq.

$$\sum_{i=o}^{n} (a_i - b_i)\, d^i k(z)/dz^i = 0 \quad , \tag{11}$$

where

$$b_i = \sum_{h=o}^{n-1-i} \lambda a_{i+1+h} K^{(h)}(0) \; , \quad i = 0, 1, 2, \ldots, n-1 \; ; \quad b_n = 0 \quad .$$

The boundary values satisfied by k(z) are determined by successive differentiations of equation (7). We thus get,

$$\begin{aligned} k(0) &= K(0) \quad , \\ k'(0) &= K'(0) + \lambda K(0)\, k(0) \quad , \\ k''(0) &= K''(0) + \lambda[K(0)\, k'(0) + K'(0)\, k(0)] \quad , \\ k'''(0) &= K'''(0) + \lambda[K(0)\, k''(0) + K'(0)\, k'(0) + K''(0)\, k(0)] \quad , \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{12}$$

Example: Given the kernel $K(z) = A \cos z + B \sin z$, determine the resolvent.

Since K(z) satisfies the equation,

$$d^2K/dz^2 + K(z) = 0 \quad ,$$

the resolvent, from (11) and (12) will satisfy the following differential system,

$$d^2k/dz^2 + \lambda A\, dk/dz + (1 - \lambda B)\, k(z) = 0 \quad ,$$

$$k(0) = A \quad , \quad k'(0) = B + \lambda A^2 \quad .$$

The solution of this equation is found to be,

$$k(z) = [(B + \lambda A^2 - C_2 A)/(C_1 - C_2)]e^{c_1 z} + [(B + \lambda A^2 - C_1 A)/(C_2 - C_1)]\, e^{c_2 z} \quad ,$$

where C_1 and C_2 are roots of the equation $C^2 + \lambda AC + (1 - \lambda B) = 0$.

For the special case where $A = 0$ and $B = 1$, this result becomes,

$$\begin{aligned} k(z) &= \sinh \sqrt{\lambda - 1}\; z \,/\sqrt{\lambda - 1} \quad , \quad \lambda > 1 \quad , \\ &= \sin \sqrt{1 - \lambda}\; z \,/\sqrt{1 - \lambda} \quad , \quad \lambda < 1 \quad , \\ &= z \quad , \qquad\qquad\quad \lambda = 1 \quad . \end{aligned}$$

Problems

1. If $K(z) = A$, show that $k(z) = Ae^{Az}$.
2. If $K(z) = A^2 z$, show that $k(z) = A \sinh Az$.
3. If $K(z) = A^3 z^2/2$, prove that the resolvent kernel is

$$k(z) = A[e^{Az} - e^{-Az/2}\cos\left\{\sqrt{3}\, Az/2\right\} - \sqrt{3}\, e^{-Az/2}\sin\left\{\sqrt{3}\, az/2\right\}]/3 \quad .$$

4. In general, if $K(z) = A^n z^{n-1}/(n-1)!$, show that the resolvent kernel is of the form,

$$k(z) = A(e^{Az} + \omega e^{\omega Az} + \omega^2 e^{\omega^2 Az} + \ldots\ldots + \omega^n e^{\omega^n Az})/n \quad ,$$

where ω is an n-th root of unity. (Evans)

3. Equations of the Closed Cycle. Before proceeding to more complicated aspects of the theory of the Volterra equation it will be of advantage to examine a special case of considerable importance. The title of this section refers to the Volterra equation whose kernel belongs to the group of functions permutable with unity, that is to say to the group of functions defined by the equation,

$$\int_x^y f(x, t) \cdot 1 \, dt = \int_x^y 1 \cdot f(t, y) \, dt \quad .*$$

If we represent the function defined by these integrals by $\phi(x,y)$ and take the derivatives with respect to y and x, we obtain:

$$\partial\phi/\partial y = \partial/\partial y \int_x^y f(x, t) \, dt = f(x, y) \quad ,$$

$$\partial\phi/\partial x = \partial/\partial x \int_x^y f(t, y) \, dt = -f(x, y) \quad .$$

Adding these equations we get,

$$\partial\phi/\partial x + \partial\phi/\partial y = 0 \quad ,$$

whose solution is

$$\phi(x, y) = \phi(x - y) \quad .$$

We thus see that the functions permutable with unity are of the form $f(x-t)$. That these functions are permutable with one another and thus form a group in the strict sense of the algebraic definition is easily proved. For this purpose let us consider the function of composition,

$$I(x, y) = \int_x^y f(x - t) \, g(t - y) \, dt \quad .$$

Transforming the variable of integration by means of the equation:

$$x - t = s - y \quad ,$$

we obtain at once,

$$I(x, y) = \int_x^y g(x - s) \, f(s - y) \, ds \quad .$$

This shows that functions permutable with unity are also permutable with one another.

One especially important sub-group is the set of functions obtained from successive powers of unity. We easily obtain the following sequence:

$$\overset{*}{1}=1; \quad \overset{*}{1}^2=(y-x)/1!; \quad \overset{*}{1}^3=(y-x)^2/2! \; ; \ldots ; \quad \overset{*}{1}^n=(y-x)^{n-1}/(n-1)! \quad .$$

*The name, the closed cycle, is derived from the fact that the group has an important application in Volterra's theory of the phenomenon of heredity in mechanical systems. For an example of a simple closed cycle phenomenon see Davis: The Present Status of Integral Equations, p. 27.

The law implied here is readily extensible to fractional values of the index and we shall have,

$$1^{p*}1^{q} = \overset{*}{1}^{p+q} \quad ; \quad \overset{*}{1}^{a} = (y-x)^{a-1}/\Gamma(a) \quad , \quad a>0 \quad .$$

An instructive application of these ideas is found in the solving of an integral equation of the form:

$$f(x) = \int_{o}^{x} K(x-t)\, u(t)\, dt \quad , \quad f(0)=f'(0)= \ldots = f^{(n-1)}(0) = 0, \qquad (13)$$

where we write,

$$K(z) = z^{-a}(a_o+a_1z+a_2z^2+ \ldots +a_nz^n) \quad , \quad 0<a<1 \quad .$$

This equation was discussed in the author's previous study where Whittaker's solution was derived by means of fractional operators.*

If we use the abbreviation $b_i = a^i\Gamma(i+1-a)$, then the kernel may be written,

$$K(z) = \sum_{i=o}^{n} b_i \ \overset{*}{1}^{i-a+1} \quad ,$$

and equation (13) appears in the following form,

$$[\overset{*}{1}^{1-a} (\sum_{i=o}^{n} b_i \overset{*}{1}^{i})]*u = f \quad . \qquad (14)$$

The solution of this equation is formally obtained by the inversion,

$$u = (\sum_{i=o}^{n} b_i \overset{*}{1}^{i})^{-1}* \overset{*}{1}^{a-1}*f \quad , \qquad (15)$$

where meaning must be given to the operator in the parenthesis.

By Lagrange's interpolation formula a function f(z) can be expressed as follows:†

$$f(z) = \frac{(z-B)\,(z-C)\ldots(z-N)}{(A-B)\,(A-C)\ldots(A-N)}f(A) + \frac{(z-A)\,(z-C)\ldots(z-N)}{(B-A)\,(B-C)\ldots(B-N)}f(B)$$
$$+ \ldots . + \frac{(z-A)\,(z-B)\ldots(z-M)}{(N-A)\,(N-B)\ldots(N-M)}f(N)$$
$$+ (z-A)\,(z-B)\ldots(z-N)\, f^{(n)}(z)/n! \quad ,$$

where A, B, ... , N are a set of numbers in the region of definition of f(z).

*The Inversion of Integrals of Volterra Type, pp. 57-60; See also reference 362.

†See Whittaker and Robinson: The Calculus of Observations, London (1924), p. 33.

If we let $f(z) = z^n$ and make the abbreviation $G(z) = \sum_{i=1}^{n} b_i z^{n-i} = b_o(z-A)(z-B)\ldots(z-N)$, where A, B, ..., N are roots of $F(z) = 0$, Lagrange's formula may be written,

$$z^n = G(z)A^n/G'(A)(z-A) + G(z)B^n/G'(B)(z-B) + \ldots\ldots + G(z)N^n/G'(N)(z-N) + G(z)/b_o \quad . \tag{16}$$

It is clear that (16) can also be written in the form,

$$z^n/G(z) = 1/G(1/z) = 1/b_o + A^n z/G'(A)(1-Az) + B^n z/G'(B)(1-Bz) + \ldots\ldots + N^n z/G'(N)(1-Nz) \quad .$$

Returning now to equation (15) and recalling the identity just written down, we can replace it symbolically by the new expansion,

$$u(x) = \overset{*}{1}{}^{a-1}[1/b_o + \left\{A^n/G'(A)\right\}\left\{\overset{*}{1}/(1^o - \overset{*}{A}1)\right\} + \left\{B^n/G'(B)\right\}\left\{\overset{*}{1}/(1^o - \overset{*}{B}1)\right\} + \ldots\ldots + \left\{N^n/G'(N)\right\}\left\{\overset{*}{1}/(1^o - \overset{*}{N}1)\right\}]*f \quad . \tag{17}$$

The interpretation of the symbol $\overset{*}{1}/(1^o - \overset{*}{A}1)$ is obtained as follows:

Let us write $\overset{*}{1}/(1^o - \overset{*}{A}1) = \phi(z)$, where $z = y - x$. From this we obtain

$$\phi - A\,1*\phi = 1 \text{ , or } \phi(z) - A\int_o^z \phi(t)\,dt = 1 \text{ .}$$

The solution of this equation is found by inspection to be $\phi(z) = e^{Az}$.

Similarly we can interpret $\overset{*}{1}{}^a*\phi(z)$ to be,

$$\overset{*}{1}{}^a*\phi(z) = \int_x^y e^{A(y-t)}(t-x)^{a-1}/\Gamma(a)\,dt \quad .$$

Making the transformation $s = t - x$, we obtain,

$$\overset{*}{1}{}^a*\phi(z) = \int_o^z e^{A(z-s)}s^{a-1}/\Gamma(a)\,ds \quad .$$

Since, moreover, $\overset{*}{1}{}^{-1}*f = df(x)/dx$, we can write (17) in the form,

$$u(x) = L*df/dx \quad ,$$

where we use the abbreviation,

$$L = \overset{*}{1}{}^a[1/b + \left\{A^n/G'(A)\right\}\left\{\overset{*}{1}/(1^o - \overset{*}{A}1)\right\} + \left\{B^n/G'(B)\right\}\left\{\overset{*}{1}/(1^o - \overset{*}{B}1)\right\} + \ldots\ldots]\ .$$

Replacing the operators in this expression by their equivalent values and writing $G(A) = \Gamma(1-a)\,F(A)$, we obtain,

$$L=[1/\Gamma(a)\Gamma(1-a)]\ [z^{a-1}/a_o + \left\{A^n/F'(A)\right\}\int_o^z e^{A(z-s)}s^{a-1}ds$$

$$+\left\{B^n/F'(B)\right\}\int_o^z e^{B(z-s)}s^{a-1}ds + \ldots\ldots\ldots] \quad .$$

Making use of the fact that $\Gamma(a)\ \Gamma(1-a) = \pi/\sin\pi a$ and that the incomplete gamma function is defined as

$$\gamma_a(z) = \int_o^z e^{z-s}s^{a-1}ds$$

we obtain Whittaker's kernel:

$$L(z) = [\sin\pi a/\pi]\ [z^{a-1}/a_o + A^{n-a}\gamma_a(Az)/F'(A) + B^{n-a}\gamma_a(Bz)/F'(B) + \ldots..]$$

4. Differential and Integral Operators of Infinite Order. A close relationship exists between integral equations and the theory of differential and integral operators of infinite order.

Let us designate by the symbol D(u) the expression,

$$D(u)=a_ou(x)+a_1du/dx+a_2(d^2u/dx^2)/2!+\ \ldots\ +a_n(d^nu/dx^n)/n!+\ \ldots$$

and by I(u) the corresponding integral expansion,

$$I(u)=a_ou(x)+a_1\int_o^x u(t)dt+a_2\int_o^x (x-t)\ u(t)dt+$$

$$a_3\int_o^x (x-t)^2\ u(t)\ dt/2!+\ \ldots\ +a_n\int_o^x (x-t)^{n-1}u(t)\ dt/(n-1)!+\ \ldots$$

The coefficients, a_n, in both expansions are functions of x.

Both operators can be symbolically represented as follows:

$$D(u)=(a_o+a_1z+a_2z^2/2!+\ \ldots\ +a_nz^n/n!+\ \ldots\ )\rightarrow u(x) \quad , \qquad (18)$$

$$I(u)=(a_o+a_1z^{-1}+a_2z^{-2}+\ \ldots\ +a_nz^{-n}+\ \ldots\ )\rightarrow u(x) \quad , \qquad (19)$$

where z and z^{-1} represent the derivative and integral operators respectively.

Designating the operators by D(x,z) and I(x,z) respectively we see that (18) and (19) can be further simplified by the notation:

$$D(u) = D(x, z)\rightarrow u(x) \quad ,$$
$$I(u) = I(x, z)\rightarrow u(x) \quad .$$

The functions D(x,z) and I(x,z) will be called respectively the *generatrix functions of first and second kind.*

The object of this and subsequent sections is to show how a symbolic calculus developed principally by S. Pincherle and C. Bourlet* is effectively employed in the solution of integral equations

*S. Pincherle: Mémoire sur le calcul fonctionnel distributif. Mathematische Annalen, vol. 49 (1897), pp. 325-381; C. Bourlet: Sur les opérations en général et les équations différentielles linéaires d'ordre infini. Annales de l'école normale supérieure, 3rd series, vol. 14 (1897), pp. 133-190; also Sur certaines équations analogues aux équations différentielles. Annales de l'école normale supérieure, 3rd series, vol. 16 (1899), pp. 333-375.

of Volterra type. This calculus is more general, within the range of functions to which it applies, than the application which we make here would indicate, since it may be used in the solution of both linear differential and linear difference equations.

By the degree (Stufe) of an unlimitedly differentiable function we mean the value L defined as the limit,*

$$L = \lim_{n=\infty} \left\{ |f^{(n)}(x)| \right\}^{1/n} .$$

It is clear that L is finite for entire functions of genus less than or equal to one, but infinite for functions possessing poles, branch points, or essential singularities in the finite plane.

Limiting our discussion to functions of finite degree we can readily deduce the following properties of L:

1. If f(x) is an analytic function of degree L, then L is independent of x.
2. If f(x) is of degree L, then $f^{(m)}(x)$ is of degree L.
3. If $f_1(x)$ and $f_2(x)$ are of degree L_1 and L_2 respectively, where $L_2 > L_1$, then the degree of $a_1f_1(x) + a_2f_2(x)$ will not exceed the degree of L_2.
4. If f(x) is of degree L, then f(ax) is of degree |a| L.
5. If $f_1(x)$ is of degree L_1 and $f_2(x)$ is of degree L_2, then $f(x) = f_1(x)f_2(x)$ is at most of degree $L_1 + L_2$.
6. If $u_1(x), u_2(x), \ldots, u_n(x), \ldots$ are functions of degrees $L_1, L_2, \ldots, L_n, \ldots$ which are bounded with n, then the function,

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x) ,$$

is of degree not greater than the upper bound of the L_n provided $\sum_{n=1}^{\infty} |a_n|$ converges.

7. If u(x) is a function of degree L, then $f(x) = \sum_{n=1}^{\infty} a_n u^{(n)}(x)$ is of degree not greater than L provided $\sum_{n=1}^{\infty} |a_n| L^n$ converges.

8. If u(x) is a function of degree $L<1$ and if $a_n(x)$, n = 1, 2, 3, ... , are functions of degree not greater than M, then the function $f(x) = \sum_{n=1}^{\infty} a_n(x) u^{(n)}(x)$ is of degree not greater than L + M provided $\Sigma|a_n(x)|$ converges.†

*This limit has been variously called *degree*, *grade*, and *exponential value*.

†For the statement of some of these theorems see O. Perron: Lineare Differentialgleichungen unendlich hoher Ordnung mit ganzen rationalen Koeffizienten. Mathematische Annalen, vol. 84 (1921), pp. 31-32.

By analogy, the definition of the degree of an umlimitedly integrable function would be,

$$M = \lim_{n=\infty} \left\{ |f^{(-n}(x)| \right\}^{1/n}.$$

If f(x) is a function integrable and bounded in an interval (ab), $|f(x)| \leqq F$, we derive the important conclusion,

$$M = \lim_{n=\infty} \left\{ |f^{(-n)}(x)| \right\}^{1/n} = \lim_{n=\infty} \left\{ \left| \int_a^x \left\{ (x-t)^{n-1} f(t)/(n-1)! \right\} dt \right| \right\}^{1/n},$$

$$\leqq \lim_{n=\infty} \left\{ \int_a^x \left\{ (x-t)^{n-1} F/(n-1)! \right\} dt \right\}^{1/n} \leqq \lim_{n=\infty} (b-a) \left\{ F/(n!) \right\}^{1/n} = 0.$$

It is for this reason that integral operators are, in general, amenable to a simpler discipline than are the corresponding differential operators of infinite order.

Definition: If S(x,z) and T(x,z) are operators of first or second kind, then the symbolic product [S(x,z) · T(x,z)] denotes the generatrix obtained when S(x,z) operates upon T(x,z). For brevity we shall frequently indicate this product by [S · T], following the notation of Bourlet, or occasionally by S → T.

We first prove the following fundamental theorem:

Theorem 4. If S(x,z) and T(x,z) are operators of first or of second kind, then the symbolic product $[S \cdot T] = [S(x,z) \cdot T(x,z)]$ *may be expanded as follows:*

$$[S \cdot T] = S \cdot T + (\partial T/\partial x)(\partial S/\partial z) + (\partial^2 T/\partial x^2)(\partial^2 S/\partial z^2)/2! + \ldots$$
$$+ (\partial^n T/\partial x^n)(\partial^n S/\partial z^n)/n! + \ldots \quad (20)$$

To prove this we proceed formally from the following well known generalization of the Leibnitz formula in elementary calculus:*

$$F(x, z) \to uv = vF(x, z) \to u + (v'/1!)\partial F(x, z)/\partial z \to u +$$
$$(v''/2!)\partial F(x, z)/\partial z \to u + \ldots (v^{(n)}/n!)\partial^n F(x, z)/\partial z^n \to u + \ldots \quad (21)$$

By definition:

$$[S \cdot T] \to u = S(x, z) \to T(x, z) \to u .$$

Let us assume first that S and T are operators of first kind and that both have Taylor's expansions about the origin, which we can denote by,

$$S(x, z) = S_0(x) + S_1(x)z + S_2(x)z^2 + \ldots ,$$
$$T(x, z) = T_0(x) + T_1(x)z + T_2(x)z^2 + \ldots .$$

*See S. Pincherle: Funktionaloperationen und Gleichungen, Encyklopädie der mathematischen Wissenschaften, II, A, 11, p. 769; also Encyclopédie des sciences mathématiques, II, 5, 26, p. 10.

It is then clear that we shall have,

$$[S \cdot T] \to u = S(x, z) \to [T_0(x)u(x) + T_1(x)u'(x) + T_2(x)u''(x) + \ldots\ldots] \ ,$$

or making use of the Leibnitz formula:

$$[S \cdot T] \to u = \sum_{j=o}^{\infty} \left\{ T_o^{(j)}(x)/j! \right\} \left\{ \partial^j S(x, z)/\partial z^j \right\} \to u$$

$$+ \sum_{j=o}^{\infty} \left\{ T_1^{(j)}(x)/j! \right\} \left\{ \partial^j S(x, z)/\partial z^j \right\} \to u' + \sum_{j=o}^{\infty} \left\{ T_2^{(j)}(x)/j! \right\} \left\{ \partial^j S(x, z)/\partial z^j \right\} \to u''$$

$$+ \ldots\ldots = \sum_{i=o}^{\infty} \sum_{j=o}^{\infty} \left\{ T_i^{(j)}/j! \right\} \left\{ \partial^j S(x, z)/\partial z^j \right\} \to u^{(i)} \ .$$

Designating the differentiation of u by placing z^i on the left side of the arrow, we obtain:

$$[S \cdot T] \to u = \sum_{i=o}^{\infty} \sum_{j=o}^{\infty} z^i \left\{ T_i^{(j)}/j! \right\} \left\{ \partial^j S(x, z)/\partial z^j \right\} \to u \ ,$$

$$= \sum_{j=o}^{\infty} \left\{ \sum_{i=o}^{\infty} z^i T_i^{(j)} \right\} \left\{ \partial^j S(x, z)/\partial z^j \right\} /j! \to u \ ,$$

$$= \sum_{j=o}^{\infty} \left\{ \partial^j T(x, z)/\partial x^j \right\} \left\{ \partial^j S(x, z)/\partial z^j \right\} /j! \to u \ ,$$

$$= S(x, z)\, T(x, z) + (\partial S/\partial z)(\partial T/\partial x) + (\partial^2 S/\partial z^2)(\partial^2 T/\partial x^2)/2! + \ldots\ldots \ .$$

That the same derivation applies to integral operators also will be apparent from a consideration of the following expansion:

$$z^{-n}(uv) = \int_o^x \ldots \int_o^t (uv)dt = \int_o^x \left\{ (x-t)^{n-1} uv/(n-1)! \right\} dt \ ,$$

$$= \int_o^x \left\{ (x-t)^{n-1}/(n-1)! \right\} u(t) \left\{ v(x) - (x-t)v'(x) + (x-t)^2 v''(x)/2! - \ldots \right\} dt,$$

$$= v(x)z^{-n}(u) - nv'(x)z^{-n-1}(u) + n(n+1)v''(x)z^{-n-2}(u)/2!$$
$$- n(n+1)\,(n+2)v'''(x)z^{-n-3}(u)/3! + \ldots\ldots \ .$$

Since this is equivalent to the Leibnitz formula applied to z^{-n}, it is clear that the expansion $I \to (uv)$ will also follow the rule, subject to proper conditions of convergence. The derivation of the formula for the symbolic product $[S \cdot T] \to u$, where S and T are integral operators, is then seen to follow identically the one given above.

Let us now examine the linear functional equation,

$$F(x, z) \to u = f(x) \ , \tag{22}$$

where F(x,z) is a known operator of first or second kind and f(x) is a given function which satisfies appropriate conditions of differentiability. It is significant to observe that these conditions are

often essentially unnecessary since they may appear only in the symbolic process and vanish in the result.

For example, the integral

$$\int_o^x u(t)dt = {}_oD_x^{-1}u = z^{-1} \rightarrow u$$

may be written in equivalent form as,

$$\begin{aligned}\int_o^x u(t)dt &= \int_o^x \left\{u(x)+(t-x)u'(x)+(t-x)u''(x)/2! + \ldots .\right\} dt \quad , \\ &= x\,u(x) - x^2u'(x)/2! + x^3u''(x)/3! - \ldots \quad , \\ &= \left\{x - x^2z/2! + x^3z^2/3! - x^4z^3/4! + \ldots .\right\} \rightarrow u(x) \quad , \\ &= (1-e^{-xz})/z \rightarrow u(x) \quad .\end{aligned}$$

We thus have the symbolic identity,

$$z^{-1} \equiv (1-e^{-xz})/z \quad .$$

But it will be observed that, in general, weaker conditions are to be imposed upon a function operated on by the left-hand member than upon a function operated on by the right-hand member. In the first case we need assume only integrability; in the second case the function must be infinitely differentiable.

It can be proved by a similar expansion of terms that the following symbolic identities hold within the range of definition of the functions operated upon:

$$z^{-2} \equiv \left\{1-(1+xz)e^{-xz}\right\}/z^2 \quad ,$$

$$z^{-3} \equiv \left\{1-(1+xz+x^2z^2/2!)e^{-xz}\right\}/z^3 \quad ,$$

and, in general,

$$\begin{aligned}z^{-n} &\equiv \left\{1-[1+xz+x^2z^2/2! + \ldots + x^{n-1}z^{n-1}/(n-1)!]e^{-xz}\right\}/z^n \quad , \\ &\equiv \left\{x^n/(n-1)!\right\}\left\{A_n - A_{n+1}xz + A_{n+2}x^2z^2 + \ldots\right\} \quad , \qquad (23)\end{aligned}$$

where we use the abbreviation $A_{n+r} = 1/(n+r)r!$.

If the lower limit of the integral is "*a*" instead of zero, x is to be replaced by x—a.

The application of these ideas to the solution of functional equations in general, and of integral equations of Volterra type in particular, is made as follows:

Returning to a consideration of equation (22), the problem is to find the *resolvent generatrix*, X(x,z), so that the solution of (22) can be represented symbolically in the form:

$$u(x) = X(x, z) \rightarrow f(x) \quad . \qquad (24)$$

If we operate on both sides of (22) with X(x,z), we get,

$$[X \cdot F] \rightarrow u(x) = X(x, z) \rightarrow f(x) \quad .$$

In order to obtain (24) it is necessary and sufficient for us to assume that X(x,z) is a solution of the *generatrix equation*,

$$[X \cdot F] = 1 \quad ,$$

or more explicitly from (20),

$$X \cdot F + (\partial X/\partial z)\,(\partial F/\partial x) + (\partial^2 X/\partial z^2)\,(\partial^2 F/\partial x^2)/2! + \ldots = 1 \quad . \tag{25}$$

Hence the problem of solving equation (22) is transformed to the problem of finding a particular solution of equation (25).

5. The Solution of Integral Equations by the Method of Generatrix Functions. The solution of the Volterra equation of second kind with an analytic kernel can be obtained in a useful form by means of the operational methods discussed in the preceding section.

Let us assume that the kernel of the equation,

$$a_0(x)\,u(x) + \int_a^x K(x, t)\,u(t)\,dt = f(x) \quad , \tag{26}$$

can be expanded in a Taylor's series as follows:

$$K(x, t) = a_1(x) + a_2(x)\,(x-t) + a_3(x)\,(x-t)^2/2! + a_4(x)\,(x-t)^3/3! + \ldots \quad .$$

From the fact that

$$z^{-m} \rightarrow u(x) = \int_a^x \left\{ (x-t)^{m-1} u(t)/(m-1)! \right\} dt \quad ,$$

it is seen that (26) can be written in the form:

$$\left\{ a_0(x) + a_1(x) z^{-1} + a_2(x) z^{-2} + a_3(x) z^{-3} + \ldots \right\} \rightarrow u(x) = f(x) \quad . \tag{27}$$

Operating upon both sides of this equation successively by z^{-1}, z^{-2}, etc., and applying the rule of Leibnitz (21), we obtain the following infinite set of linear equations:

$$\left\{ a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots \right\} \rightarrow u(x) = f(x) \quad ,$$

$$\left\{ a_0 z^{-1} + (a_1 - a_0')\, z^{-2} + (a_2 - a_1' + a_0'')\, z^{-3} + \ldots \right\} \rightarrow u(x) = z^{-1} \rightarrow f(x) \quad ,$$

$$\left\{ a_0 z^{-2} + (a_1 - 2a_0')\, z^{-3} + (a_2 - 2a_1' + 3a_0'')\, z^{-4} + \ldots \right\} \rightarrow u(x) = z^{-2} \rightarrow f(x) \quad ,$$

$$\left\{ a_0 z^{-3} + (a_1 - 3a_0')\, z^{-4} + (a_2 - 3a_1' + 6a_0'')\, z^{-5} + \ldots \right\} \rightarrow u(x) = z^{-3} \rightarrow f(x)$$

. . . . . .

$$\left\{ a_0 z^{-n} + (a_1 - n a_0')\, z^{-n-1} + [a_2 - n a_1' + n(n+1) a_0''/2!]\, z^{-n-2} + [a_3 - n a_2' + n(n+1) a_1''/2! - n(n+1)(n+2) a_0'''/3!]\, z^{-n-3} + \ldots \right\} \rightarrow u(x) = z^{-n} \rightarrow f(x)$$

. . . . . .

Solving this system of equations for u(x) we obtain the following expansion:

$$u(x)=\left\{b_0+b_1z^{-1}+b_2z^{-2}+b_3z^{-3}+\ .\ .\ .+b_mz^{-m}+\ .\ .\ .\right\}\rightarrow f(x)\quad ,\qquad (28)$$

where we use the abbreviations:

$$b_0=1/a_0,\ b_1=-a_1/a_0^2,\ b_2=\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1-a_0' \end{vmatrix}/a_0^3,\ b_3=\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1-a_0' & a_2-a_1'+a_0'' \\ 0 & a_0 & a_1-2a_0' \end{vmatrix}/a_0^4\quad ,$$

$$\ldots,\quad b_m=\begin{vmatrix} a_1 & a_2 & a_3 & \ldots. & a_m \\ a_0, & a_1-a_0', & a_2-a_1'+a_0'', & \ldots, & a_{m-1}-a_{m-2}'+\ldots\mp a_0^{(m-1)} \\ 0 & a_0 & a_1-2a_0' & \ldots. & a_{m-2}-2a_{m-3}'+\ldots\mp(m-1)a_0^{(m-2)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots. & a_1-(m-1)a_0' \end{vmatrix}/a_0^{m+1}.$$

Let us now prove that the function,

$$X(x,\ z)=b_0+b_1z^{-1}+b_2z^{-2}+b_3z^{-3}+\ .\ .\ .\ .\ .\ ,\qquad (29)$$

is in fact a formal solution of the generatrix equation.

Forming the product $[X\cdot F]$ we get,

$$[X\cdot F]=\left\{b_0+b_1z^{-1}+b_2z^{-2}+\ldots.\right\}\left\{a_0+a_1z^{-1}+a_2z^{-2}+\ldots.\right\}$$
$$-z^{-2}\left\{b_1+2b_2z^{-1}+3b_3z^{-2}+\ldots.\right\}\left\{a_0'+a_1'z^{-1}+a_2'z^{-2}+\ldots.\right\}$$
$$+z^{-3}\left\{2b_1+2\cdot 3b_2z^{-1}+3\cdot 4b_3z^{-2}+\ldots\right\}\left\{a_0''+a_1''z^{-1}+a_2''z^{-2}+\ldots.\right\}/2!$$
$$-\ldots\ldots\ldots\equiv a_0b_0+[a_1b_0+a_0b_1]z^{-1}+[a_2b_0+(a_1-a_0')b_1+a_0b_2]z^{-2}$$
$$+[a_3b_0+(a_2-a_1'+a_0'')b_1+(a_1-2a_0')b_2+a_0b_3]z^{-3}+\ldots\ .$$

Equating this expansion to unity we obtain the following set of equations:

$$a_0b_0=1\quad ,$$
$$a_1b_0+a_0b_1=0\quad ,$$
$$a_2b_0+(a_1-a_0')b_1+a_0b_2=0\quad ,$$
$$a_3b_0+(a_2-a_1'+a_0'')b_1+(a_1-2a_0')b_2+a_0b_3=0\quad ,$$
$$\cdot\qquad\cdot\qquad\cdot\qquad\cdot\qquad\cdot\qquad\cdot\qquad\cdot$$

From this system we can determine the values of $b_0,\ b_1,\ b_2,\ \ldots.$ and the expansion thus obtained is found to coincide with (28).

Moreover the solution is unique provided all the $a_i(x)$ exist at the point $x=0$ and $a_0(0) \neq 0$. To prove this statement assume that X is of the form,

$$X(x, z) = A(x)e^{-xz} + X_0(x, z) \quad ,$$

where $X_o(x,z)$ is the function already obtained. Substituting this in the generatrix equation and recalling that

$$a_i(0) = a_i(x) - x\ a_i'(x) + x^2\ a_i''(x)/2! - \ldots \quad ,$$

we get,

$$[X \cdot F] = 1 + A(x)e^{-xz}\left\{a_0(0) + a_1(0)z^{-1} + a_2(0)z^{-2} + \ldots\right\} \quad , \qquad (30)$$

where A(x) is an arbitrary function.

Since we have

$$e^{-xz} \rightarrow \left\{\phi(z) \rightarrow u(x)\right\} = \left\{e^{-xz}\phi(z)\right\} \rightarrow u(x) \quad ,$$

which follows at once from the generatrix product (20), it is clear that (30) will furnish a resolvent generatrix for equation (26) provided

$$e^{-xz} \rightarrow \left\{[a_0(0) + a_1(0)z^{-1} + a_2(0)z^{-2} + \ldots] \rightarrow f(x)\right\}$$

is identically zero for all functions f(x) suitably defined. It is obvious from the operational identity

$$e^{-xz} \rightarrow g(x) = g(0) \quad ,$$

that the expression in the brackets must vanish at $x=0$ if (30) is to be a resolvent generatrix.

But from the fact that,

$$\left\{a_0(0) + a_1(0)z^{-1} + a_2(0)z^{-2} + \ldots\right\} \rightarrow f(x) = a_0(0)f(x) + \int_0^x K(x-t)f(t)dt \quad ,$$

it follows that we shall have

$$e^{-xz} \rightarrow \left\{a_0(0)f(x) + \int_0^x K(x-t)f(t)dt\right\} = a_0(0)f(0) \quad ,$$

which is zero for all functions f(x) only when $a_0(0) = 0$. Since this case is explicitly excluded by the conditions imposed, the uniqueness of the operator is thus established.

We thus prove the theorem:

Theorem 5. If $F(x,z) = a_0(x) + a_1(x)z^{-1} + a_2(x)z^{-2} + \ldots\ldots$, *where the* $a_i(x)$ *are functions analytic in a common region about the*

origin and where $a_0(0) \neq 0$, is an integral operator of infinite order, then the resolvent generatrix $X(x,z)$ satisfying the equation $[X \cdot F] = 1$, is also an integral operator of infinite order and has the expansion given by equation (28). This operator is also unique.

It is interesting to notice the special case which includes the theory of the group of the closed cycle discussed in section 3. If the $a_i(x)$ are assumed to be constants we have the case of the integral equation,

$$u(x) + \int_0^x K(x-t)\, u(t)\, dt = f(x) \quad , \tag{31}$$

where

$$K(x-t) = K(0) + (x-t)K'(0) + (x-t)^2 K''(0)/2! + \ldots ; \quad a_0 = 1,$$
$$a_i = K^{(i-1)}(0), \quad i \neq 0 \quad .$$

Clearly the resolvent generatrix will be,

$$X(x, z) = 1/F(x, z) = \left\{1 + K(0)/z + K'(0)/z^2 + K''(0)/z^3 + \ldots\right\}^{-1} \quad ,$$

and we have the following theorem:

Theorem 6. The resolvent kernel for the integral equation of the closed cycle,

$$u(x) + \int_0^x K(x-t)\, u(t)\, dt = f(x) \quad ,$$

is formally equivalent to the expansion,

$$k(x-t) = k(0) + k'(0)\ (x-t) + k''(0)\ (x-t)^2/2! + \ldots\ldots \quad ,$$

where the $k^{(i)}(0)$ are determined from the equation,

$$\left\{1 + K(0)r + K'(0)r^2 + K''(0)r^3 + \ldots\right\}^{-1} =$$
$$1 + k(0)r + k'(0)r^2 + k''(0)r^3 + \ldots\ldots \quad .$$

The solution thus becomes,

$$u(x) + f(x) + \int_0^x k(x-t)\, f(t)\, dt \quad .$$

A few examples will illustrate the application of this theorem.

Example 1. Solve the equation

$$u(x) + \int_0^x e^{(x-t)} u(t)\, dt = f(x) \quad .$$

From the resolvent generatrix,

$$X = 1/(1 + 1/z^2 + \ldots) = 1 - 1/z \quad ,$$

the solution is immediately obtained:

$$u(x) = f(x) - \int_0^x f(t)\, dt \quad .$$

Example 2. Consider the equation

$$u(x) + \lambda \int_0^x (x-t)\, u(t)\, dt = f(x)$$

Since the resolvent generatrix is,

$$X = 1/(1+\lambda/z^2) = 1 - \lambda/z^2 + \lambda^2/z^4 - \lambda^3/z^6 + \ldots\ldots,$$

the solution will be,

$$u(x) = f(x) - \lambda \int_0^x \left\{ (x-t) - \lambda (x-t)^3/3! + \lambda^2 (x-t)^5/5! + \ldots \right\} f(t)\, dt \quad ,$$

$$= f(x) + \sqrt{\lambda} \int_0^x \sin \sqrt{\lambda}\, (t-x)\, f(t)\, dt \quad .$$

Example 3. (T. Kameda, reference 470, p. 202).

Solve the equation,

$$u(x) + \int_0^x \psi(x-t)\, u(t)\, dt = f(x) \quad ,$$

where the kernel is of the form,

$$\psi(z) = 1 + z/1!\,2! + z^2/2!\,3! + z^3/3!\,4! + \ldots\ldots\ldots \quad ,$$
$$= -i\, J_1(2i\sqrt{z})/\sqrt{z} \quad ,$$

where $J_1(x)$ is the Bessel function of first order.

Computing the resolvent generatrix we get,

$$X = \left\{ 1 + \sum_{n=0}^{\infty} \psi^{(n)}(0) z^{n+1} \right\}^{-1} = e^{-z} = 1 - z + z^2/2! - \ldots\ldots\ldots \quad .$$

Hence the resolvent kernel will be,

$$k(z) = -1 + z/1!\,2! + z^2/2!\,3! - \ldots\ldots\ldots = -\psi(-z) \quad .$$

The solution of the integral equation thus becomes,

$$u(x) + \int_0^x \psi(t-x)\, f(t)\, dt = f(x) \quad .$$

6. An Application of the Generatrix Functions of Abel to the Equation of the Closed Cycle. An alternative way of deriving the result stated in theorem 6 of the last section has been given by T. Kameda, who employed the generatrix functions of Abel. We shall give a brief résumé of this method.*

*Reference 470.

The integral,

$$J(f)=\int_{-\infty}^{\infty} e^{-xz}f(x)dx \quad ,$$

regarded as a function of the parameter z is called the *generatrix function* of f(x), where f(x) is regarded as limited and integrable in every finite interval. The function f(x) is known as the *determinant function.*

If

$$\int_{-\infty}^{\infty} |f(x)e^{-xz}|dx$$

converges for all values of z in a region G, and if I(x) is defined as the integral,

$$I(x)=\int_{-\infty}^{x} (x-t)^{p-1}f(t)dt/\Gamma(p),\ p>0 \quad , \tag{32}$$

then we shall have $J(I)=J(f)/z^p$ in the region common to G and the region $R(z)>0$, where R(z) represents the real part of z.

By definition,

$$J(I)=\int_{-\infty}^{\infty} e^{-zx}\left\{\int_{-\infty}^{x} (x-t)^{p-1}f(t)dt/\Gamma(p)\right\}dx$$

$$=\int_{-\infty}^{\infty} f(t)dt\int_{t}^{\infty} (x-t)^{p-1}e^{-zx}dx/\Gamma(p) \quad .$$

Letting $x-t=r/z$, we have

$$J(I)=\int_{-\infty}^{\infty} e^{-zt}f(t)dt\int_{0}^{\infty} r^{p-1}e^{-r}dr/z^p\Gamma(p) \quad ,$$

$$=\int_{-\infty}^{\infty} e^{-zt}f(t)dt/z^p=J(f)/z^p \quad .$$

Returning to the main proposition, namely the proof of theorem 6, we shall first establish the following equation:

$$J\left\{\int_{0}^{x}K(x-t)g(t)dt\right\}=\left\{\sum_{n=0}^{\infty}K^{(n)}(0)/z^{n+1}\right\}J(g) \quad .$$

Let us begin with the observation that,

$$\int_{0}^{x}K(x-t)g(t)dt=\int_{0}^{x}\sum_{n=0}^{\infty}K^{(n)}(0)(x-t)^n\left\{g(t)/n!\right\}dt=$$

$$\sum_{n=0}^{\infty}\int_{0}^{x}K^{(n)}(0)(x-t)^n\left\{g(t)/n!\right\}dt,$$

provided the sum

$$\sum_{n=o}^{\infty} K^{(n)}(0)(x-t)^n g(t)/n!$$

converges uniformly within a finite interval of x.

Operating with J we have,

$$J\left\{\int_o^x K(x-t)g(t)dt\right\} = \int_{-\infty}^{\infty} e^{-xz}\int_o^x K(x-t)g(t)dt\,dx =$$

$$\int_{-\infty}^{\infty} e^{-xz}\sum_{n=o}^{\infty}\int_o^x K^{(n)}(0)(x-t)^n\left\{g(t)/n!\right\}dtdx =$$

$$\sum_{n=o}^{\infty} K^{(n)}(0)\int_{-\infty}^{\infty} e^{-xz}\int_o^x (x-t)^n\left\{g(t)/n!\right\}dtdx \quad .$$

The last step is justified provided the integrals exist. This assumption will be met by assuming that g(t) is bounded within an interval (0b) and is identically zero outside.

Hence we have

$$\sum_{n=o}^{\infty} K^{(n)}(0)\int_{-\infty}^{\infty} e^{-xz}\int_o^x (x-t)^n\left\{g(t)/n!\right\}dtdx =$$

$$\sum_{n=o}^{\infty} K^{(n)}(0)\int_{-\infty}^{\infty} e^{-zx}\int_{-\infty}^{x} (x-t)^n\left\{g(t)/n!\right\}dtdx = J(g)\sum_{n=o}^{\infty} K^{(n)}(0)/z^{n+1} \quad .$$

Operating upon equation (31) with J and making use of the result just obtained we get,

$$J(u)+J(u)\sum_{n=o}^{\infty} K^{(n)}(0)/z^{n+1} = J(f) \quad ,$$

or

$$J(u) = \left\{1+\sum_{n=o}^{\infty} K^{(n)}(0)/z^{n+1}\right\}^{-1} J(f) \quad ,$$

$$= \left\{1+k(0)/z+k'(0)/z^2+k''(0)/z^3+ \ldots .\right\} J(f) \quad .$$

Making the hypothesis that f(x) is bounded in the interval (0b) and is identically zero outside, this equation reduces to,

$$J(u) = J(f)+J\left\{\int_o^x k(x-t)f(t)dt\right\} \quad ,$$

or because of the linearity of the operator J,

$$J\left\{u-f-\int_o^x k(x-t)f(t)dt\right\} = 0 \quad . \qquad (33)$$

We now make use of the following theorem due to M. Lerch:*

If the definite integral

$$J(z)=\int_0^\infty e^{-xz}f(x)dx$$

corresponding to an integrable continuous or discontinuous determinant function f(x), exists for a certain value of z it exists for every larger value. It can not vanish for an infinity of positive values of z which form an arithmetic sequence unless one has $J(z)\equiv 0$ and, in general, $f(x)\equiv 0$.

Applying this theorem to equation (33) it is clear that the determinant function must be identically zero and hence we get,

$$u(x)=f(x)+\int_0^x k(x-t)f(t)dt \quad .$$

Problems

1. Prove that

$$z^{-1}+z^{-2}+z^{-3}+\ldots\ldots\equiv(e^x e^{-zx}-1)/(1-z) \quad .$$

2. Prove the symbolic identity:

$$-\left\{z^{-1}/x+2!z^{-2}/x^2+3!z^{-3}/x^3+\ldots\ldots\right\}=1-xz\ e^{-xz}\int_{-\infty}^{xz}(e^t/t)dt \quad .$$

3. Prove that the solution of

$$u(x)=f(x)+\int_0^x K(x-t)u(t)dt \quad ,$$

where $K(z)=K_0+K_1z+K_2z^2/2!+K_3z^3/3!+\ldots\ldots$ and $f(x)=f_0+f_1x+f_2x^2/2!+f_3x^3/3!+\ldots\ldots$, is given by $u(x)=u_0+u_1x+u_2x^2/2!+u_3x^3/3!+\ldots\ldots$, where $u_0=f_0$, $u_1=f_1+K_0u_0$, $u_2=f_2+K_0u_1+K_1u_0$, $\ldots\ldots$, $u_r=f_r+K_0u_{r-1}+K_1u_{r-2}+\ldots\ldots+K_{r-1}u_0$. (Gorakh Prasad, reference 509.)

4. Prove that the solution of

$$u(x)=f(x)+\int_0^x K(x,t)u(t)dt \quad ,$$

where $K(x,t)=K_{00}+K_{10}x+K_{01}t+(K_{20}x^2+2K_{11}xt+K_{02}t^2)/2!+(K_{30}x^3+3K_{21}x^2t+3K_{12}xt^2+K_{03}t^3)/3!+\ldots\ldots$, and $f(x)=f_0+f_1x+f_2x^2/2!+f_3x^3/3!+\ldots\ldots$, is given by $u(x)=u_0+u_1x+u_2x^2/2!+u_3x^3/3!+\ldots\ldots$, where $u_0=f_0$, $u_1=f_1+K_{00}u_0$, $u_2=f_2+(2K_{10}+K_{01})u_0+K_{00}u_1$, $u_3=f_3+(3K_{20}+3K_{11}+K_{02})u_0+(3K_{10}+2K_{01})u_1+K_{00}u_2$, $\ldots\ldots$, and, in general, $u_r=f_r+\Sigma K_{r-p-q-1,p}\,u_q\,r!/(r-p-q-1)!p!q!(p+q+1)!$. The summation, Σ, is to be extended over all

*Acta Mathematica, vol. 27 (1903), pp. 339-351; in particular p. 347.

positive integral values of p and q including 0 which satisfy the inequality $r-p-q-1 \geqq 0$. (Gorakh Prasad, reference 509.)

5. Prove that the solution of the equation $u(x)=f(x)+\int_0^x K(x-t)u(t)dt$ is $u(x)=(1/2\pi i)\int_{a-i\infty}^{a+i\infty}\left\{e^{xz}F(z)/[1-k(z)]\right\}dz$, where $F(z)=\int_0^\infty e^{-xz}f(x)dx$, and $k(z)=\int_0^\infty e^{-xz}K(x)dx$. (V. A. Fock, reference 582.)

7. A Second Solution of Integral Equations by the Method of the Resolvent Generatrix. The solution of integral equations of Volterra type given in section 2 can be obtained as a special case of the general solution of the generatrix equation,

$$[X \cdot F]=1 \quad , \tag{34}$$

where $F(x,z)$ is the generatrix function corresponding to the integral equation.

If we assume that $F(x,z)$ is of the form,

$$F(x, z) = 1 - \lambda g(x, z) \quad , \tag{35}$$

then the following function,

$$X(x, z)=1+\lambda g(x, z)+\lambda^2 g^{(2)}(x, z)+\lambda^3 g^{(3)}(x, z)+ \ldots . \quad , \tag{36}$$

where $g^{(m)}(x,z)$ means the generatrix of the n-th power of the operator, is easily seen to be a formal solution of the equation.

To prove this we substitute (35) and (36) in equation (34) and thus obtain,

$$[X \cdot F]\equiv 1+\lambda(g-g)+\lambda^2[g^{(2)}-g^2-(\partial g/\partial x)(\partial g/\partial z)-(\partial^2 g/\partial x^2)(\partial^2 g/\partial z^2)/2!- \ldots .]+\lambda^3[g^{(3)}-g\,g^{(2)}-(\partial g/\partial x)(\partial g^{(2)}/\partial z)-(\partial^2 g/\partial x^2)(\partial^2 g^{(2)}/\partial z^2)/2!- \ldots .]+ \ldots .$$

Since we have by definition,

$$g^{(m)}(x, z)=g\,g^{(m-1)}+(\partial g/\partial x)(\partial g^{(m-1)}/\partial z)+(\partial^2 g/\partial x^2)(\partial^2 g^{(m-1)}/\partial z^2)/2!+ \ldots$$

the multiplier of λ^m is seen to vanish for every value of m and (36) is thus the formal solution of equation (34).

In order to identify the solution $u(x)=X(x,z) \rightarrow f(x)$ with that of Volterra we observe that,

$$g^{(2)}(x, z) \rightarrow f(x)=\int_0^x K(x, t)\int_0^t K(x, s)f(s)dsdt=$$

$$\int_0^x f(s)ds\int_s^x K(x, t)K(t, s)dt=\int_0^x K_2(x, s)f(s)ds \quad ,$$

and, in general,

$$g^{(m)}(x, z) \to f(x) = \int_0^x K_m(x, s) f(s) ds \quad ,$$

where $K_m(x,s)$ is the m-th iterated kernel. This fact identifies the solution just obtained with that of section 2.

8. The Equation with Kernel of the form $K(x, t) = g(x,t)/(x-t)^a$, $a<1$. In sections 1 and 2 it was shown that for continuous kernels and for a finite range of the independent variables, the Volterra integral equation of second kind has, in general, one unique solution and the homogeneous equation has no solution other than the trivial one $u = 0$. In this section we shall consider the Volterra equation of the form,

$$u(x) = f(x) + \lambda \int_A^x [g(x, t)/(x-t)^a] \; u(t) dt \quad , \quad a<1 \quad , \tag{37}$$

where $f(x)$ is a function continuous in the range $A \leqq x \leqq B$ and $g(x,t)$ is continuous over the region $A \leqq t \leqq x \leqq B$. *The number a is called the index of the kernel.*

Employing the abbreviation, $S(u) = \int_A^x K(x, t)\, u(t)\, dt$, we have previously proved that the integral equations,

$$u(x) = f(x) + \lambda S(u) \quad ,$$

and

$$u(x) = f(x) + \lambda S(f) + \lambda^2 S^{(2)}(f) + \lambda^3 S^{(3)}(f) + \ldots + \lambda^{n-1} S^{(n-1)}(f) + \lambda^n S^{(n)}(u)$$

$$= f(x) + \lambda \int_A^x \Big\{ K(x, t) + \lambda K_2(x, t) + \lambda^2 K_3(x, t) + \ldots$$

$$+ \lambda^{n-2} K_{n-1}(x, t) \Big\} f(t) dt + \lambda^n \int_A^x K_n(x, t)\, u(t) dt \quad , \tag{38}$$

where $K_i(x,t)$ is the i-th iterated kernel, are essentially equivalent since the solution of each is unique and the solution of one is the solution of the other.

We now wish to show that, n being taken sufficiently large, the index of the kernel $K_n(x,t)$ is no longer positive.

To prove this let us consider the function of composition formed from the functions $f_1(x,y) = g_1(x,y)/(x-y)^a$ and $f_2(x,y) = g_2(x,y)/(x-y)^b$, where a and b are less than one and where $g_1(x,y)$ and $g_2(x,y)$ are continuous functions over the range considered. We thus get

$$f_3(x, y) = \int_y^x \Big\{ g_1(x, t) g_2(t, y)/(x-t)^a (t-y)^b \Big\} dt \quad .$$

If we let G_1 and G_2 represent the upper limits of the absolute values of the two functions $g_1(x,t)$ and $g_2(t,y)$ and if we change variables from t to s according to the equation $t = x+s(y-x)$, it is clear that $f_3(x,y)$ may be bounded as follows:

$$|f_3(x,y)| \leqq G_1G_2/(x-y)^{a+b-1}\int_0^1 \left\{1/s^a(1-s)^b\right\} ds \quad ,$$

$$= G_1G_2\Gamma(1-a)\Gamma(1-b)/(x-y)^{a+b-1}\,\Gamma(2-a-b) \quad .$$

The integral is clearly finite provided a and b are less than 1. *Hence the index of $f_3(x,y)$ is not greater than $a+b-1$.*

We can apply this criterion immediately to $K_n(x,y)$ when the original kernel is equal to $g(x,y)/(x-y)^a$, $a<1$. It follows that the index of $K_2(x,y)$ is at most equal to $2a-1$, that of $K_3(x,y)$ is at most equal to $3a-2$, and that of $K_p(x,y)$ at most equal to $pa-(p-1)$. Hence, in order to determine a kernel which will be finite for all values of x and y in the region of definition of $g(x,y)$, it is sufficient to choose $n > 1/(1-a)$. That this kernel is also continuous is readily proved from the assumptions.

Thus we see that the existence of a unique solution for equation (38) follows at once from the general existence theorem already established and this equation carries with it the existence theorem for equation (37).

The actual solving of the equation treated in this section can be facilitated in many cases by the use of the theory of fractional operators. This theory was developed in considerable detail in the author's study on the inversion of Volterra integrals to which the reader is referred for a more extended treatment. Fractional integration was there defined to be,*

$${}_cD_x^{-a}u(x) = \int_c^x \left\{(x-t)^{a-1}\,u(t)/\Gamma(a)\right\} dt \quad , \quad a>0,$$

and fractional differentiation by the equation,

$${}_cD_x^{m+a}u(x) = (d^{m+1}/dx^{m+1})\int_0^x \left\{(x-t)^{-a}u(t)/\Gamma(1-a)\right\} dt \quad ,$$

$$m=0,\ 1,\ 2,\ \ldots\ ;\quad 0<a<1 \quad .$$

These symbols were shown to satisfy the following linearity conditions:

$${}_cD_x^{\mu}\left\{u(x)+v(x)\right\} = {}_cD_x^{\mu}u(x) + {}_cD_x^{\mu}v(x) \quad ,$$

*J. D. Tamarkin has recently discussed these operators when they are applied to Lebesgue integrable functions. See reference 597.

$$ {}_cD_x^{\mu} k\, u(x) = k\, {}_cD_x^{\mu} u(x) \quad , $$

$$ {}_cD_x^{\mu}\, {}_cD_x^{\nu} u(x) = {}_cD_x^{\mu+\nu} u(x) \quad . $$

When there is no question of ambiguity the symbolic notation $z^{-a} \rightarrow u(x)$ is often useful to designate ${}_cD_x^{-a}\, u(x)$.

By methods analogous to those previously given in the former study it can be shown that the integral equation,

$$ P(z) = (A_0 + A_1 z^{-1/n} + A_2 z^{-2/n} + \ldots + A_{n-1} z^{1/n-1}) \rightarrow u(x) = f(x) \quad , $$

can be formally replaced by the integral equation,

$$ A(z) \rightarrow u(x) = \triangle(z) \rightarrow f(x) \quad , \tag{39} $$

where we use the abbreviation,

$$ \triangle(z) = \begin{vmatrix} 1 & A_1 & A_2 & \ldots\ldots & A_{n-1} \\ z^{-1/n} & A_0 & A_1 & \ldots\ldots & A_{n-2} \\ z^{-2/n} & z^{-1}A_{n-1} & A_0 & \ldots\ldots & A_{n-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ z^{1/n-1} & z^{-1}A_2 & z^{-1}A_3 & \ldots\ldots & A_0 \end{vmatrix} \quad , $$

and $A(z)$ is the same determinant with the first column replaced by $A_0 , z^{-1} A_{-1} , z^{-1} A_{n-2} , \ldots . , z^{-1} A_1$.

Since $A(z)$ is a polynomial in z^{-1} of degree less than or equal to n—1, equation (39) is an integral equation from which the discontinuity of the kernel has been removed.

The following example illustrates the application.*

Consider the equation,

$$ u(x) + \int_0^x \left\{ u(t)/(x-t)^{1/3} \right\} dt = f(x) \quad . \tag{40} $$

This equation may be written in the form,

$$ \left\{ 1 + \Gamma(2/3) z^{-2/3} \right\} \rightarrow u(x) = f(x) \quad . $$

Obviously we have,

$$ A(z) = \begin{vmatrix} 1 & 0 & \Gamma(2/3) \\ z^{-1}\Gamma(2/3) & 1 & 0 \\ 0 & z^{-1}\Gamma(2/3) & 1 \end{vmatrix} = 1 + z^{-2}\Gamma^3(2/3) \quad , $$

and similarly

$$ \triangle(z) = 1 - \Gamma(2/3) z^{-2/3} + \Gamma^2(2/3) z^{-4/3} \quad . $$

*See reference, 79, p. 104.

Hence equation (40) is equivalent to the *rationalized* equation,

$$u(x) + \Gamma^3(2/3)\int_0^x (x-t)u(t)dt = F(x) \quad ,$$

in which

$$F(x) = f(x) + \int_0^x \left\{-(x-t)^{-1/3} + \Gamma^2(2/3)(x-t)^{1/3}/\Gamma(4/3)\right\} f(t)\, dt \quad .$$

The solution of this equation is

$$u(x) = F(x) + \lambda \int_0^x \sin \lambda\,(t-x)\, F(t)\, dt \quad ,$$

where $\lambda = \Gamma^{3/2}(2/3)$. This function may be shown to be a solution of (40) also.

It is of interest to exhibit in comparison the method of generatrix functions employed by Kameda (section 7) and the method of fractional operators. For this purpose let us consider the equation,

$$u(x) + \left\{1/\Gamma(1/2)\right\} \int_0^x (x-t)^{-1/2} u(t)\, dt = f(x) \quad .$$

Operating with J and recalling (32) we obtain,

$$\begin{aligned} J(u) + J(u)/z^{1/2} &= J(f) \quad , \\ J(u) &= (1+1/z^{1/2})^{-1} J(f) \quad , \\ &= (1-1/z^{1/2}+1/z-1/z^{3/2}+1/z^2 - \ldots)\, J(f) \quad . \end{aligned}$$

Since from equation (32)

$$J(f)/z^a = J\left\{\int_0^x (x-t)^{a-1} f(t)\, dt/\Gamma(a)\right\} \quad ,$$

we get,

$$J(u) = J\left\{f(x) + \int_0^x e^{x-t} f(t)\, dt - \int_0^x [(x-t)^{-1/2}/\Gamma(1/2)\right.$$

$$\left. + (x-t)^{1/2}/\Gamma(3/2) + (x-t)^{3/2}/\Gamma(5/2) + \ldots] f(t)\, dt\right\} \quad .$$

Hence, making use of the fact that $\Gamma(1/2) = \sqrt{\pi}$, the solution becomes,

$$u(x) = f(x) + \int_0^x e^{x-t} f(t)\, dt - (1/\sqrt{\pi}) \int_0^x (x-t)^{-1/2} S(x-t)\, f(t)\, dt \quad ,$$

where

$$S(z) = 1 + 2z + (2z)^2/1 \cdot 3 + (2z)^3/1 \cdot 3 \cdot 5 + \ldots\ldots \quad .$$

Turning to the method of fractional integrals we write the equation in the form,

$$(1 + z^{-1/2}) \rightarrow u(x) = f(x) \quad .$$

A simple calculation shows that this rationalizes into the equation,

$$u(x) - \int_0^x u(t)\, dt = f(x) - {}_0D_x^{-1/2} f(x) \quad .$$

The resolvent for the kernel $K(x,t) = -1$ is $k(x,t) = e^{x-t}$, which leads with a simple transformation to Kameda's solution.

A similar rationalization can be applied to an equation of the form,

$$u(x) + \int_0^x g(t)\ (x-t)^{a-1} u(t)\, dt = f(x), \quad a > 0 \quad , \tag{41}$$

where g(t) is a polynomial of n-th degree.

This equation may be written in symbolic form as follows:

$$u(x) + \Gamma(a) \left\{ g(x)\, u(x) \right\}^{(-a)} = f(x) \quad .$$

Applying the rule of Leibnitz (21) to the expression $\left\{ g(x)\, u(x) \right\}^{(-a)} = z^{-a} \rightarrow g(x)\, u(x)$, we can expand (41) into the equation:

$$u(x) + \Gamma(a) \left\{ g(x) u^{(-a)}(x) - a g'(x) u^{(-a-1)}(x) + a(a+1)\, g''(x)\, u^{(-a-2)}(x)/2! - \ldots + (-1)^n a(a+1) \ldots (a+n-1)\, g^{(n)}(x) u^{(-a-n)}(x)/n! \right\} = f(x) \quad . \tag{42}$$

Expressing the generatrix of the left-hand member of this equation in the form,

$$1 + \Gamma(a) \left\{ g z^{-a} - a g' z^{-a-1} + a(a+1) g'' z^{-a-2}/2! - \ldots + (-1)^n a(a+1) \ldots (a+n-1) g^{(n)} z^{-a-n}/n! \right\} \quad ,$$

we can operate upon this generatrix with z^{-b} and obtain by use of the rule of Leibnitz (21), the expansion:

$$z^{-b} + \Gamma(a) \left\{ g\, z^{-a-b} - C_1\, g'\, z^{-a-b-1} + C_2\, g''\, z^{-a-b-2}/2! - C_3\, g'''\, z^{-a-b-3}/3! - \ldots + (-1)^n\, C_n\, g^{(n)}\, z^{-a-b-n}/n! \right\} \quad ,$$

where we use the abbreviations,

$$C_1 = a + b,\ C_2 = (a+b)\ (a+b+1),\ C_3 = (a+b)\ (a+b+1)\ (a+b+2), \ldots \ldots \quad .$$

If $a = 1/m$ and we let b equal successively the values $1/m,\ 1/m+1,\ \ldots,\ 1/m+n;\ 2/m,\ 2/m+1,\ \ldots,\ 2/m+2n;\ 3/m,\ 3/m+1,\ \ldots\ldots,\ 3/m+3n;\ \ldots\ (m-1)/m,\ (m-1)/m+1,\ (m-1)/m+2,\ \ldots,\ (m-1)/m+(m-1)n$, then these generatrix functions operating upon (42) will

yield a set of equations from which the fractional operators may be removed by setting up the eliminant of Sylvester's dialytic method.*

Two examples will illustrate the process of rationalization described above.

Example 1. Let us rationalize the equation,

$$u(x) + \int_0^x t(x-t)^{-1/2}\, u(t)\, dt = f(x) \quad .$$

Writing this in symbolic form we have,

$$u(x) + \Gamma(1/2)\ (ux)^{(-1/2)} = f(x) \quad ,$$

or

$$u(x) + \Gamma(1/2)\ (xu^{(-1/2)} - u^{(-3/2)}/2) = f(x) \quad .$$

Operating with $z^{-1/2}$ and $z^{-3/2}$ we get,

$$u^{-(1/2)} + \Gamma(1/2)\ (xu^{(-1)} - u^{(-2)}) = f^{(-1/2)}\ (x) \quad ,$$

$$u^{(-3/2)} + \Gamma(1/2)\ (xu^{(-2)} - 2u^{(-3)}) = f^{(-3/2)}\ (x) \quad .$$

Eliminating $u^{(-1/2)}$ and $u^{(-3/2)}$ from the three equations thus obtained and noting that $\Gamma(1/2) = \sqrt{\pi}$ we obtain the rationalized equation:

$$u(x) - \pi \int_0^x \left\{ t(x+t)/2 \right\} u(t)dt = f - \sqrt{\pi}\, x\, f^{(-1/2)} + \sqrt{\pi}\, f^{(-3/2)}/2 \quad .$$

Example 2. Given the equation,

$$u(x) + \int_0^x t^2\ (x-t)^{-2/3} u(t)dt = f(x) \quad ,$$

we may write it symbolically in the form,

$$u(x) + \Gamma(1/3)\ (x^2u)^{(-1/3)} = f(x) \quad .$$

Operating successively by the proper fractional integrals we obtain the following set of equations:

$$u(x) + \Gamma(1/3) \left\{ x^2u^{(-1/3)} - 2xu^{(-4/3)}/3 + 4u^{(-7/3)}/9 \right\} = f(x) \quad ,$$

$$u^{(-1/3)} + \Gamma(1/3) \left\{ x^2u^{(-2/3)} - 4xu^{(-5/3)}/3 + 10u^{(-8/3)}/9 \right\} = f^{(-1/3)}(x) \quad ,$$

$$u^{(-4/3)} + \Gamma(1/3) \left\{ x^2u^{(-5/3)} - 10xu^{(-8/3)}/3 + 40u^{(-11/3)}/9 \right\} = f^{(-4/3)}(x) \quad ,$$

$$u^{(-7/3)} + \Gamma(1/3) \left\{ x^2u^{(-8/3)} - 16xu^{(-11/3)}/3 + 88u^{(-14/3)}/9 \right\} = f^{(-7/3)}(x) \quad ,$$

$$u^{(-2/3)} + \Gamma(1/3) \left\{ x^2u^{(-1)} - 2xu^{(-2)} + 2u^{(-3)} \right\} = f^{(-2/3)}(x) \quad ,$$

$$u^{(-5/3)} + \Gamma(1/3) \left\{ x^2u^{(-2)} - 4xu^{(-3)} + 6u^{(-4)} \right\} = f^{(-5/3)}(x) \quad ,$$

$$u^{(-8/3)} + \Gamma(1/3) \left\{ x^2u^{(-3)} - 6xu^{(-4)} + 12u^{(-5)} \right\} = f^{(-8/3)}(x) \quad ,$$

*See M. Bôcher: Introduction to Higher Algebra (1921), pp. 198-200.

$$u^{(-11/3)} + \Gamma(1/3)\left\{x^2u^{(-4)} - 8xu^{(-5)} + 20u^{(-6)}\right\} = f^{(-11/3)}(x) \quad ,$$

$$u^{(-14/3)} + \Gamma(1/3)\left\{x^2u^{(-5)} - 10xu^{(-6)} + 30u^{(-7)}\right\} = f^{(-14/3)}(x) \quad .$$

Elimination of all fractional integrals from this system yields the equation:

$$u(x) + \Gamma^3(1/3)\int_0^x t^2(44x^4 + 40x^3t + 75x^2t^2 + 40xt^3 + 44t^4)\, u(t)\, dt/243 = f - \Gamma(1/3)\,(x^2\, f^{(-1/3)} - 2x\, f^{(-4/3)}/3 + 4\, f^{(-7/3)}/9) + \Gamma^2(1/3)\,(x^4\, f^{(-2/3)} - 2x^3\, f^{(-5/3)} + 34\, x^2 f^{(-8/3)}/9 - 16\, x\, f^{(-11/3)}/3 + 352\, f^{(-14/3)}/81) \quad .$$

PART II

THE SINGULAR INTEGRAL EQUATION

1. Preliminary Survey of the Problem. In the following sections we shall consider the Volterra integral equation,

$$u(x) = f(x) + \int_a^x K(x,t)\, u(t)dt \quad , \tag{1}$$

for the two singular cases: (1) where the lower limit of the integral is infinite and (2) where the kernel has discontinuities of a type different from those previously considered, i.e. where $K(x,t) = g(x,t)/(x-t)^a$, $a<1$, $g(x,t)$ continuous.

M. Bôcher has shown the necessity of imposing the condition of continuity upon the kernel if a unique, continuous, non-trivial solution of the Volterra equation is to be obtained by citing the example:*

$$u(x) = f(x) + \int_0^x t^{x-t} u(t)dt \quad .$$

It is easily seen that the solution of this equation will be $u(x) = u_o(x) + kx^{x-1}$, where $u_o(x)$ is a particular solution and k is an arbitrary parameter.

Another example which exhibits some of the difficulties to be encountered in this subject is furnished by the following equation:

$$u(x) = \int_0^x \left\{(x-t)/x^4\right\} u(t)dt \quad . \tag{2}$$

Making the transformation: $x = 1/y$, $t = 1/s$, and the change of dependent variable: $u(1/y) = e^{-y} y^3 v(y)$, we obtain the equivalent equation:

$$v(y) = \int_y^\infty e^{y-s} (s-y)\, v(s)\, ds \quad . \tag{3}$$

This is clearly a closed cycle equation and in spite of its singularity is amenable to treatment by methods previously explained. Let us assume that $v(s)$ is expansible in a Taylor series:

$$v(s) = v(y) + (s-y)\, v'(y) + (s-y)^2\, v''(y)/2! + \ldots . \tag{4}$$

Substituting in the equation and noting that

$$\int_y^\infty e^{y-s} (s-y)^n\, ds = \int_0^\infty e^{-t} t^n dt = n! \quad ,$$

*Reference 5, pp. 16-17.

we replace the integral equation by the following differential equation of infinite order:

$$v(y) = v(y) + 2v'(y) + 3v''(y) + \ldots\ldots \quad ,$$

the only solution of which is $v(y) = A$. Thus we obtain as the solution of (2) the function $u(x) = Ae^{-1/x}/x^3$. The existence of an essential singularity in this solution indicates the nature of the difficulties experienced in founding an adequate theory of the singular equation.*

The author has discussed at some length in his study on the inversion of integrals the nature of the singularities appearing in integral equations of first kind which are analogous to the singularities in linear differential equations of Fuchsian type and in differential equations with essential singularities. This problem has been treated directly for the Volterra integral equation of second kind in much detail by M. Watanabe.† His discussion, however, will be omitted from the present study, because it will be evident that the theorems which applied in the case of equations of first kind with analytic kernels carry over without essential change to equations of the second kind.

Thus let us suppose that the integral equation to be studied appears in the form:

$$\phi(x)\, u(x) + \int_a^x G(x,t)\, u(t)dt = g(x), \quad \phi(a) = 0. \tag{5}$$

Computing two new functions,

$$K(x,t) = \int_t^x G(x,t)dx + \phi(t), \quad f(x) = \int_a^x g(x)dx \quad ,$$

we observe that equation (1) is, in general, equivalent to the following Volterra equation of first kind,

$$\int_a^x K(x,t)\, u(t)dt = f(x), \tag{6}$$

since we can pass from (6) to (1) by taking the derivative of (6).

For this reason in this brief survey of a difficult topic we shall pursue a different line of investigation which will supplement the

*Similar examples are furnished by E. Goursat who shows that the solution of $u(x) = \lambda \int_x^\infty \left\{ (s-x)^n/n! \right\} u(s)ds$ is $u(x) = Ae^{-ax}$ where $a = (\lambda)^{1/(n+1)}$ and that the solution of $u(x) = \lambda \int_0^x (1/s - 1/x)\, u(s)ds$ is $u(x) = Ax^a$ where $a^2 + a = \lambda$. References 439 and 6, p. 434.

†Reference 96.

methods of Holmgren, Lalesco, Horn, and others previously followed. In this discussion the approach will be made thru the contributions of G. C. Evans and C. E. Love.

It will be clear at the outset that case (1) where the lower limit of the integral is infinite, is included in case (2) where the kernel is infinitely discontinuous, since the transformation: $x = a + 1/y$, $t = a + 1/s$, $u(a + 1/y) = \phi(y)$, changes equation (1) into the equation:

$$\phi(y) = f(a+1/y) - \int_{\infty}^{y} \left\{K(a+1/y,\ a+1/s)/s^2\right\}\phi(s)ds \quad .$$

Similarly the transformation: $x = 1/(y-a)$, $t = 1/(s-a)$, $u\{1/(y-a)\} = v(y)$, changes the equation,

$$u(x) = f(x) + \int_{\infty}^{x} K(x,t)\ u(t)dt \quad ,$$

into

$$v(y) = f\left\{1/(y-a)\right\} - \int_{a}^{y} [K\left\{1/(y-a),\ 1/(s-a)\right\}/(s-a)^2]\ v(s)ds \quad .$$

By means of such transformations we are able to examine an integral equation for spurious singularities. Thus let us consider the equation,

$$u(x) = f(x) + \int_{x}^{\infty} \left\{G(x,t)\ u(t)/g(t)\ h(x)\right\}\ dt \quad ,$$

where $f(x)$, $g(x)$, $h(x)$, and $G(x,t)$ are known functions subject to conditions which will be specified later.

The transformation $v(x) = f(x)/h(x)$ reduces the problem to a consideration of the equation,

$$v(x) = h(x)\ f(x) + \int_{x}^{\infty} \left\{G(x,t)\ v(t)/g(t)\ h(t)\right\}\ dt \quad .$$

Hence it will be sufficient to examine the equation,

$$u(x) = f(x) + \int_{x}^{\infty} \left\{G(x,t)\ u(t)/g(t)\right\}\ dt \quad . \tag{7}$$

With the equation in this form we then make the transformation: $x = 1/y$, $t = 1/s$ and thus obtain,

$$u(1/y) = f(1/y) + \int_{0}^{y} \left\{G\ (1/y, 1/s)/s^2 g\ (1/s)\right\} u(1/s)ds \quad .$$

What conditions must now be imposed upon the known functions in order to bring the equation under the circumstances of Theorem 1,

Part I? The following theorem, due to Evans, is readily seen to furnish an answer to this question and reveals the spurious nature of the apparent singularity of the equation:*

Theorem 1. If the following conditions are fulfilled: (1) $f(x)$ *continuous in an interval* $x \geqq a$; (2) $|f(x)| < M$, *where* M *is finite, for* $x \geqq a$; (3) $\lim_{x=\infty} f(x)$ *exists;* (4) $g(x)$ *is continuous and positive for* $x > 0$; (5) $\lim_{x=\infty} \int_a^x dt/g(t)$ *exists;* (6) $G(x,t)$ *continuous and less than* M *in the region* $t \geqq x \geqq a$; *then there exists one and only one solution of equation* (7) *which is finite and continuous thruout the interval* $x \geqq a$.

2. The Equation with Integrable Kernel. In order to enlarge the scope of the problem we now introduce the following four conditions:

(*A*) A real function of the two variables (x,t) is to be continuous in a triangle S: $a \leqq t \leqq x \leqq b$, $b > a \geqq 0$, except on a finite number of curves, each composed of a finite number of continuous pieces with continuously turning tangents. The vertical portions are to be finite in number, $x = \alpha_1, \alpha_2, \ldots, \alpha_k$, and each of the curves, exclusive of these vertical portions, is to possess only a finite number of vertical tangents. Hence, except at the points $x = \alpha_1, \alpha_2, \ldots, \alpha_k$, each curve will be cut by a line $x = x_0$ at only a finite number of points.

If this condition is fulfilled we shall say that the discontinuities of the function are *regularly distributed.*†

(*B*) In the interval σ: $a \leqq x \leqq b$, a real function of the variable x is to be continuous except at a finite number of points, $x = \beta_1, \beta_2, \ldots, \beta_l$, and is to remain finite.

(*C*) A function $\int_a^x |K(x,t)|dt$ converges for all values of x in the interval $a \leqq x \leqq b$ except over a finite set $x = \lambda_1, \lambda_2, \ldots,$ and λ_m remains less than M where M is finite.

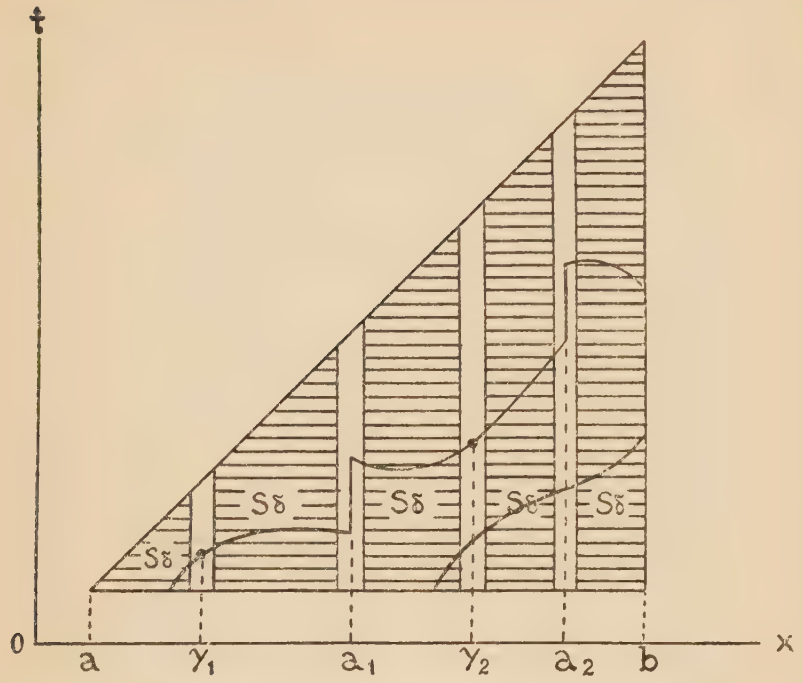
(*D*) If ε and δ are two arbitrarily chosen positive quantities then there is a quantity η and a region S_δ obtained by removing from S the small strips $\delta_i - \delta < x < \delta_i + \delta$, $i = 1, 2, \ldots, n$, n finite, such that the following inequality holds,

$$\int_y^{y+\eta} |K(x,t)|\, dt < \varepsilon \ ,$$

when (x,y) and $(x, y+\eta)$ are in S_δ.

*Reference 436, p. 411.

†This definition is a slight extension of the definition originally given by Bôcher. Reference 5, p. 2.



The following theorem due to Evans may then be stated:*

Theorem 2. If the kernel of equation (1) exists in a region S subject to conditions (A), (C), and (D), where the points $x = \delta_1, \delta_2, \ldots, \delta_n$ *include the sets* $x = \alpha_1, \alpha_2, \ldots, \alpha_k$ *and* $x = Y_1, Y_2, \ldots, Y_m$, *and if* $f(x)$ *exists in the range* σ *subject to condition (B), and if, moreover, the range* σ *can be divided into p parts bounded by the points* $x = a = a_0, a_1, a_2, \ldots, b = a_p$, *such that the following inequality holds:*

$$\int_{a_i}^{x} |K(x, t)| \, dt \leqq H < 1, \quad \alpha_i \leqq x \leqq \alpha_{i+1}, \quad x \neq a_1, a_2, \ldots, a_k, \tag{8}$$

then there exists one and only one solution $u(x)$ *of equation (1) continuous in* σ *except for a finite number of points included in the sets* $x = \{\beta_i\}$ *and* $x = \{\delta_i\}$.

In proving this theorem we shall need to establish the following points:

(*a*) That the function $u_{n+1}(x)$,

$$u_{n+1}(x) = \int_a^x K(x,t) \int_a^t K(t,t_1) \ldots \int_a^{t_{n-1}} K(t_{n-1}, t_n) \, f(t_n) \, dt \, dt_1 \ldots dt_n,$$

is continuous in σ_δ for all values of n.

*Reference 438, first paper.

(*b*) That the series,

$$U(x) = u_1(x) + u_2(x) + \ldots + u_{n+1}(x) + \ldots \ ,$$

is uniformly convergent in the region σ_δ and represents in σ a function finite and continuous except for a finite number of points.

(*c*) That the function,

$$u(x) = f(x) + U(x) \ ,$$

is a solution of the integral equation.

(*d*) That this solution is unique.

In establishing (*a*) let us consider the function,

$$v(x) = \int_a^x K(x, t)\, \phi\,(t)\, dt \ ,$$

where $\phi(x)$ is continuous in σ with the exceptions already noted and less in absolute value than N.

Letting $x = m$ be any point in σ_δ we form the difference,

$$|\Delta v| = |v(m+\Delta x) - v(m)| \leqq \left|\int_x^{m+\Delta x} K(m+\Delta x,t)\phi(t)dt\right| +$$

$$\left|\int_a^m \Big\{ K(m+\Delta x,t) - K(m,t) \Big\}\phi(t)dt\right|,$$

$$\leqq N\int_m^{m+\Delta x} |K(m+\Delta x,\, t)|\ |dt| +$$

$$N\int_a^m |K(m+\Delta x,\, t) - K(m,\, t)|\ |dt| \ .$$

On each side of the points $t_1, t_2, \ldots, t_r$ in which the line $x = m$ is cut by the curves of discontinuity of the kernel we lay off the distance $\eta/2$ such that,

$$\int_{t_i-\eta/2}^{t_i+\eta/2} |K(x,\, t)|\ |dt| < \varepsilon \ .$$

This can be done by virtue of hypothesis (*D*).

If we choose ε equal to $\rho/N(r+1)$, where ρ is an arbitrarily small preassigned value and choose $\Delta x < \eta$ sufficiently small so that none of the curves of discontinuity cuts the lines $t = t_i \pm \eta/2$, we shall have the inequality,

$$|\Delta v| \leqq \rho/2 + N\,\Sigma\int_{A_i}^{B_i} |K(m+\Delta x,t) - K(m,t)|\ |dt| \ ,$$

where the integration extends over the section of the line $x = m$ from which the r parts have been removed.

Because of the continuity of $K(x,t)$ in these closed regions we can choose Δx sufficiently small so that the following inequality holds,

$$|K(m+\Delta x,t) - K(m,t)| \leqq \rho/2(m-a)N \quad .$$

Substituting this inequality above we get,

$$|\Delta v| \leqq \rho/2 + N \Sigma(B_i - A_i)\ \rho/2(m-a)N \leqq \rho \quad .$$

This establishes the continuity of $u_1(x)$. But since $u_2(x) = \int_a^x K(x,t)\ u_1(t)dt$ and since $u_1(x)$ satisfies the condition assumed for $\phi(x)$, it is clear that $u_2(x)$ is also continuous in σ_δ. By mathematical induction we thus prove the continuity of $u_{n+1}(x)$.

In establishing (*b*) we derive a majorant for $u_{n+1}(x)$ in each of the $i = 1, 2, 3, \ldots, p$ intervals into which the range σ_δ has been divided by the points $a_1, a_2, a_3, \ldots, a_p$.

The proposition to be proved is that, for x in any of the first i-intervals, the following inequality holds:

$$|u_{n+1}(x)| \leqq N \left\{ (n+1)M \right\}^{i-1} H^{n+2-i} \quad . \tag{9}$$

Obviously from the assumptions of the theorem, first, that $\int_{a_i}^x |K(x,t)|dt \leqq H$, and, second, that $|f(x)| \leqq N$, we have for $a \leqq x \leqq a_1$,

$$|u_{n+1}(x)| \leqq NH \quad , \tag{10}$$

which is the statement of the inequality (9) when $i = 1$.

Also from condition (*C*) it follows that for every interval,

$$|u_{n+1}(x)| \leqq NM^{n+1} \quad , \quad M \geqq H \quad . \tag{11}$$

Again from (11) and the fact that $N\{(n+1)M\}^{i-1}H^{n+2-i} = N(n+1)^{i-1}M^{n+1}(M/H)^{i-2-n}$, we have the inequality (9) holding in the i-th interval when $n \leqq i-2$.

In order to prove (9) in general we shall use mathematical induction assuming that (9) holds in the (i—1)st interval for any value of n and then showing that it holds in the i-th interval as well.

If we adopt the abbreviation,

$$u_{n+1}[x]=N\int_a^x|K(x,t)|\int_a^t|K(t,t_1)|\ldots\int_a^{t_{n-1}}|K(t_{n-1},t_n)|dt_ndt_{n-1}\ldots dt_1dt \quad ,$$

it will be clear that, we can write the inequality,

$$|u_{n+1}(x)|\leqq u_{n+1}[x]=$$

$$\int_a^{a_{i-1}}|K(x,t)|u_n[t]dt+\int_{a_{i-1}}^x|K(x,t)|\int_a^{a_1-1}|K(t,t_1)|u_{n-1}[t_1]dt_1dt+$$

$$\int_{a_{i-1}}^x|K(x,t)|\int_{a_{i-1}}^t|K(t,t_1)|\int_a^{a_{i-1}}|K(t_1,t_2)|u_{n-2}[t_2]dt_2dt_1dt+\ldots\ldots \qquad (12)$$

$$+\int_{a_{i-1}}^x|K(x,t)|\int_{a_{i-1}}^t|K(t,t_1)|\int_{a_{i-1}}^{t_1}|K(t_1,t_2)|\ldots\ldots\ldots$$

$$\int_{a_{i-1}}^{t_{n+1-1}}|K(t_{n+1-i},t_{n+2-i})|u_{i-1}[t_{n+2-i}]dt_{n+2-i}dt_{n+1-i}\ldots dt_1dt \quad .$$

From (11) and (8) it appears that the last term of this series does not exceed $NM^{i-1}H^{n+2-i}$. Moreover we see that in all the terms excepting the last the variables t_j in $u_{n-j}[t_j]$ lie within the first i—1 intervals. Hence, by assumption, we get $u_{n-j}[t_j]\leqq N\{(n-j)M\}^{i-2}H^{n-j+2-i}$. Furthermore, making use of the inequalities

$$\int_{a_{i-1}}^{t_m}|K(t_m,t_n)|dt_n\leqq H,\ a_{i-1}\leqq t_m\leqq a_i,\ \text{and}\ \int_a^{a_{j-1}}|K(t_m,t_n)|dt_n\leqq M \quad ,$$

we may replace inequality (12) by the series,

$$|u_{n+1}(x)|\leqq NM^{i-1}H^{n+2-i}\left\{n^{i-2}+(n-1)^{i-2}+(n-2)^{i-2}+\ldots+(i-1)^{i-2}+1\right\}$$
$$\leqq NM^{i-1}H^{n+2-1}(n+1)n^{i-2}\leqq NM^{i-1}H^{n+2-1}(n+1)^{i-1} \quad ,$$

which is the desired inequality (9). We have thus proved that the inequality holds for the i-th interval and hence, by induction, for all intervals.

In this manner we establish for any point within the p intervals of σ_δ the inequality,

$$|u_{n+1}(x)|\leqq N[(n+1)M]^{p-1}H^{n+2-p}=A_n \quad .$$

Since $\lim_{n=\infty}A_{n+1}/A_n=H<1$, the series $\sum_{n=0}^{\infty}A_n$ converges and forms a majorant for $U(x)=\sum_{n=1}^{\infty}u_n(x)$ for x in σ_δ.

In establishing (*c*) and (*d*) of the outline of the proof sketched above an argument in no essential way different from that given in Section 1, Part I, is employed and need not be repeated here.*

An example suggested by Evans as an illustration of the scope of the theorem is given by the following equation:

$$u(x)=f(x)+\int_a^x \{c\ u(t)/(x-a)\}\,dt,\ c>0\ .$$

Obviously we have $\int_a^x |K(x,t)|dt = c$. If $c \geqq 1$, the equation has the solution $u(x) = A(x-a)^{c-1} + (x-a)^{c-1}\int_a^x (t-a)^{-1}\frac{d}{dt}\{(t-a)\,f(t)\}\,dt$

which is not unique and is continuous only if restrictions are imposed upon f(x) sufficient to insure the existence and continuity of the integral. If, however, $c < 1$, then A must be chosen equal to zero and under the restrictions of the theorem the term involving the integral exists and is continuous.

Another example suggested by Evans includes the kernel:

$$K(x,t) = \begin{cases} 1/5(x-a) & ,\ a \leqq x \leqq m, \\ 1/5(m-a) & ,\ a \leqq t \leqq m, \\ 1/5(m-a) + 1/2(x-m), & m < t \leqq x, \end{cases} \qquad m < x \leqq b\ .$$

The conditions of the theorem are satisfied by this kernel.

Condition (*C*) is not satisfied by the kernel,

$$K(x,t) = \begin{cases} 1/5(x-a) & , \\ 1/5(t-a) & , \end{cases} \qquad \begin{matrix} a \leqq x \leqq m, \\ m < x \leqq b\ . \end{matrix}$$

3. The Equation with Non-integrable Kernel. In the last section the case of integrable tho discontinuous kernels was considered. We shall now examine the problem presented by the

*Further generalization of the foregoing results to Lebesgue integrable functions can more properly be referred to the theory of the Fredholm integral equation since this equation includes the Volterra type when K(x,t) is assumed to be identically zero for $t \geqq x$. The introduction of a line of discontinuities along $x = t$ offers no difficulties when the known functions of (1) are assumed to be Lebesgue integrable.

This extension has been fully discussed in a recent paper by E. Hille and J. D. Tamarkin to which the reader is referred for details of treatment. (Reference 598). Because of the extensive nature of the considerations involved an adequate development of this theory cannot be given here. It is sufficient to remark that the theorems of the first and second section of Part I, both with regards to the existence of a resolvent kernel which is an entire function of λ and to the uniqueness of the solution attained by it, carry over under the generality of Lebesgue integration when the known functions of equation (1) are subject to restrictions of the following kind:

$|K(x,t)| \leqq g(t)$, where g(t) is a function integrable (Lebesgue) in the interval (0 1);

$\int_0^A g(t)\,|f(t)|\,dt < \infty$, for A arbitrarily large but fixed; $\int_0^1 |K(x,t)\,f(t)|\,dt \leqq M$.

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equation where condition (C) fails to be satisfied. For this purpose it will be sufficient to consider the equation,

$$u(x) = f(x) + \int_a^x \left\{ K(x,t)\, u(t)/k(t) \right\} dt, \tag{13}$$

where the infinite singularities are included in $1/k(t)$. We shall then assume that,

$$|K(x,t)| \leqq H < 1 ,$$

but that $\lim_{y=a} \int_y^x dt/k(t)$ does not exist.

Making the change of dependent variable $u(x) = r(x)v(x)$, equation (13) becomes,

$$r(x)\, v(x) = f(x) + \int_a^x \left\{ K(x,t)\, r(t)\, v(t)/k(t) \right\} dt \quad . \tag{14}$$

Our problem now is to make a selection of the function $r(x)$ in such a way that equation (14) can be brought under the conditions of Theorem 2. It is evident that $r(x)$ must have an essential singularity at the point $x = a$ so we take as our choice the function,

$$r(x) = e^{-\int_x^b dt/|k(t)|} .$$

Making this substitution in (14) we obtain,

$$v(x) = f(x)\, e^{\int_x^b dt/|k(t)|} + \int_a^x G(x,t)\, v(t)\, dt,$$

where the kernel is,

$$G(x,t) = K(x,t) \left\{ e^{\int_x^b dt/|f(t)| - \int_t^b dt/|f(t)|} \right\} /f(t) \quad .$$

It is easily seen that all the conditions of Theorem 2 are satisfied by this kernel. In particular we can show that $\int_a^x |G(x,t)|dt \leqq H$ since we have,

$$\int_a^x |G(x,t)|dt \leqq H \int_a^x \left[\left\{ e^{\int_x^b dt/|f(t)| - \int_t^b dt/|f(t)|} \right\} /f(t) \right] dt \tag{15}$$

Making the change of variables: $s = \int_t^b dt/|f(t)|$, $z = \int_x^b dt/|f(t)|$, and

noting that $ds = -dt/|f(t)|$; $t=a$, $s=\infty$; $t=x$, $s=z$, we find that (15) becomes $\int_z^\infty e^{z-s}ds = 1$. Hence we have $\int_a^x |G(x,t)|dt \leqq H$.

Identifying the conditions with those of Theorem 2 we may state the following result:

Theorem 3. If in equation (13) the function $K(x,t)$ satisfies the conditions of Theorem 2 and in absolute value is less than 1, if $\lim_{y=a} \int_y^x dt/k(t)$ does not exist, and if $\lim_{x=a} f(x)e^{\int_x^b dt/|k(t)|}$ remains finite, then there exists at least one solution of equation (13) finite and continuous in σ_δ.

An example is furnished by the equation,

$$u(x) = f(x) + \int_a^x \left\{ c\, u(t)/(t-a) \right\} dt \quad , \tag{16}$$

for which the formal solution is,

$$u(x) = A(x-a)^c + (x-a)^c \int_a^x (t-a)^{-c} f'(t)dt \quad . \tag{17}$$

If we let $K(x,t) = 1/m$, $m > 1$ and $k(t) = (t-a)/mc$, then $\lim_{x=a} f(x)e^{\int_x^b dt/|K(t)|} = \lim_{x=a} f(x)\ (b-a)^{m|c|}\ (x-a)^{-m|c|}$. If $c \leqq 0$ it is clear that $u(x)$ is finite provided $f(x)$ is finite at $x = a$ and A is chosen equal to zero. If $c > 0$, then there exists a one parameter family of solutions provided we have $f(x) = (x-a)^{mc} f_1(x)$, where $f_1(x)$ is finite at $x = a$.

To show that $u(x)$ furnishes a solution of the equation we substitute (17) in the right-hand member of (16). We thus get

$$f(x) + \int_a^x \left\{ cA(t-a)^{c-1} + (t-a)^{c-1} \int_a^t (s-a)^{-c} f'(s)ds \right\} dt \quad .$$

An integration by parts of the second term under the integral sign yields,

$$f(x) + \left\{ A(t-a)^c + (t-a)^c \int_a^t (s-a)^{-c} f'(s)ds \right\} \Big|_a^x - \int_a^x f'(s)ds \quad .$$

If $c > 0$, $f(x) = (x-a)^{mc} f_1(x)$, this clearly reduces to $u(x)$; if $c \leqq 0$ and $A = 0$, the expression becomes $u(x) + f(a)$. But due to

the condition that $\lim_{x=a} f(x)\ (b-a)^{m|c|}\ (x-a)^{-m|c|}$ exists, we must impose the restriction $f(a) = 0$. Hence (17) is a solution of (16) under the condition of the theorem.

PROBLEM

Show that the equation,

$$u(x) = f(x) + \int_a^x \left\{ K(x,t)\ u(t)/g(x)\ k(t) \prod_{i=1}^{\rho} [t-\psi_i(x)]^{\lambda_i} \right\} dt \quad ,$$

where $\sum_{i=1}^{\rho} \lambda_i = \lambda < 1$ and the $\psi_i(x)$ are arbitrary continuous functions, can be brought under the results of theorem 3. (Evans)

Hint: Show that this equation is equivalent to

$$v(x) = \left\{ f(x)\ g(x)/r(x) \right\} + \int_a^x \left\{ r(t)\ K(x,t)\ v(t)/g(t)\ k(t)\ r(x) \prod_{i=1}^{\rho} [t-\psi_i(x)]^{\lambda_i} \right\} dt,$$

where $r(x) = e^{-\int_x^b dt/|\alpha K(t) g(t)|^{(\nu+\lambda)/\nu}}$, $\nu > 0$, $\nu + \lambda < 1$. Choose $\alpha = 1/(1+P)$ where $P \geqq \sum_{i=1}^{\rho} \int_a^x \left\{ 1/|t - \psi_i(x)|^{\nu+\lambda} \right\} dt$.

An extension of Theorem 3 is included in the following statement of results obtained by Evans:

Theorem 4. The known functions in the equation,

$$u(x) = f(x) + \int_a^x \left\{ K(x,t)\ u(t)/k(t)\ g(x) \right\} dt, \qquad (18)$$

are assumed to satisfy the following conditions:

1. $K(x,t)$ is continuous in S and $k(x)$, $g(x)$ and their first derivatives are continuous in σ; $\partial K(x,t)/\partial x$ and $\partial K(x,t)/\partial t$ satisfy condition (A) and are finite in S; $f(x)$ is continuous in σ except at $x = a$ and the function $f(x)\ g(x)$ and its first derivative satisfy (B) in σ.

2. The function $\theta(x) = k(x)\ g(x)$ is greater than zero in σ except for the point $x = a$, where $\theta(a) = 0$ in such a way that $\lim_{y=a} \int_y^x dx/\theta(x)$ does not exist.

3. $K(a,a) \neq 0$, $\lim_{x=a} [K(x,x) - K(a,a)]/(x-a)^{\nu}$ exists, $0 < \nu < 1$, $\nu > 1 - 1/\theta'(a)$.

4. $\lim_{x=a} f(x)\ g(x) = 0$.

Under the conditions stated there exists, with the possible exception of the point $x = a$, one continuous solution of the integral equation provided $K(a,a) < 0$, and a one parameter family of solution provided $K(a,a) > 0$. In the neighborhood of $x = a$ the solution $u(x)$ remains less than $C\, f(x)/\,(x—a)^{\nu}$ where C is a constant.

Before entering upon the proof of this theorem it is useful to study a simple example which illustrates the force of the conditions imposed. For this purpose let us consider the equation,

$$u(x) = f(x) + \int_0^x \left\{K\, u(t)/t\right\} dt \quad .$$

The solution may be written explicitly,

$$u(x) = e^{-K\int_x^b dt/t} \int_R^x f'(t)\, e^{K\int_t^b dt/t}\, dt \quad ,$$

$$= x^K \int_R^x f'(t)\, t^{-K} dt \quad .$$

Assuming that there exists a value N such that $|f'(t)| < N$ for $0 \leqq t \leqq b$, we have

$$|u(x)| \leqq N\, x^K \int_R^x t^{-K} dt = \begin{cases} Nx(1—x^{K-1}R^{1-K})/(1—K), & K \neq 1 \ , \\ Nx \log (x/R) \qquad\qquad\quad , & K = 1 \ . \end{cases}$$

Since $k(x) = x/K$, we can characterize the solution in terms of $k'(0) = 1/K$.

If $k'(0) < 1$, that is to say, if $K > 1$, then $|u(x)| \leqq Cx$; if $k'(0) = 1$, then $|u(x)| \leqq Cx \cdot x^{-\nu}$ where $\nu > 0$; if $k'(0) > 1$, then $|u(x)| > Cx \cdot x^{-\nu}$, where $\nu > 1—1/k'(0) = 1—K$.

The method of proof of the theorem is suggested by the above example. We can without loss of generality (see Section 1) set $g(x)$ equal to one in equation (18). We may then write the kernel in the form,

$$K(x,t)/k(t) = K(a,a)\left\{1 + \Gamma(x,t)\right\}/k(t) \quad ,$$

where, according to the conditions of the theorem, $\lim_{x=a} \Gamma(x,t)/(x—a)^{\nu}$, $0 < \nu < 1$, $\nu > 1—1/k'(a)$, exists.

Hence, abbreviating $K(a,a)$ by K, we may write (18) in the form,

$$u(x) = f(x) + \int_a^x [K\left\{1+\Gamma(x,t)\right\} u(t)/k(t)]\, dt \quad .$$

We begin our discussion with a study of the equation,

$$u_o(x) = f(x) + \int_a^x \{K\, u(t)/k(t)\}\, dt \quad ,$$

whose explicit solution is readily seen to be,

$$u_o(x) = e^{-K\int_x^b dt/k(t)} \int_R^x f'(t) e^{K\int_t^b dt/k(t)}\, dt \quad .$$

If we employ the abbreviations:

$$s = \int_t^b dt/k(t),\ z = \int_x^b dt/k(t),\quad S_R = \int_R^b dt/k(t),$$

we can express this solution in the form,

$$u_o(x) = -e^{-Kz} \int_{s_R}^z k(t)\, f'(t)\, e^{Ks} ds \quad .$$

In order to obtain a majorant for $u_o(x)$ we consider the function

$$v_o(x) = -e^{-Kz} \int_{s_R}^z k(t)\, e^{Ks} ds \quad ,$$

which, after an integration by parts, becomes,

$$v_o(x) = -k(x)/K + k(R)\, e^{K(s_R - z)}/K + \int_R^x k'(t)\, e^{K(s-z)} dt/K \quad . \qquad (19)$$

Before we can bound $|v_o(x)|$ we must establish the following lemma:

Lemma 1. If the function k(t) is subject to the restrictions of Theorem 4 and K is positive, then the following inequality holds:

$$I(x) = e^{-K\int_x^b dt/k(t)} \leqq J\, k(x)\, (x-a)^{-1+K/k'(a)} \quad ,\quad a \leqq x \leqq b \quad ,$$

where J is a suitably chosen constant.

Let us assume first that for the interval $a \leqq x \leqq b$, we can write k(x) in the form:

$$k(x) = (x-a)\, k'(a) \{1 + a_1(x-a) + a_2(x-a)^2 + \ldots\ldots\} \quad .$$

From the limitations imposed upon k(x) by the theorem it is clearly necessary that we have $k'(a) > 0$, for otherwise k(x) would not be positive in the interval.

Substituting this value of k(x) in the integral I(x) we obtain,

$$I(x) = \{(x-a)/(b-a)\}^{K/k'(a)} \psi(x-a) \quad ,$$

where $\psi(x-a)$ is a function bounded and positive in the interval.

Hence we may write,

$$I(x) = k(x)\ (x-a)^{-1+K/k'(a)}\phi(x-a) \quad ,$$

where $\phi(x-a)$ is positive and bounded in the interval. Choosing the constant J greater than the upper bound of $\phi(x-a)$ we see that the lemma is proved for the assumption made regarding the form of k(x).

Assuming next that we can write k(x) in the form,

$$k(x) = (x-a)^{\lambda}\ k_o(x-a) \quad ,$$

where $\lambda > 1$ and $k_o(0) > 0$, we can readily show that

$$I(x) = e^{-K/k_o(0)\ (\lambda-1)\ (x-a)^{\lambda-1}}\ \phi(x-a) \quad ,$$

where $\phi(x-a)$ is a function finite and positive in the interval.

Since we have

$$\lim_{x=a} I(x)/k(x) = \lim_{x=a} (x-a)^{-\lambda}e^{-K/k_o(0)\ (\lambda-1)\ (x-a)^{\lambda-1}}\ \phi(x-a)/k_o\ (x-a) = 0 \quad ,$$

the lemma is seen to hold for this case. A similar argument is readily supplied for the assumption that k(x) vanishes more strongly than in the last case, for example, where the reciprocal of k(x) has an essential singularity at $x = a$.

Returning to equation (19) and assuming that $K > 0$, $|k'(t)| < N$, we obtain from the lemma the following inequality,

$$|v_o(x)| \leqq k(x) \left\{ 1/K+J\,k(R)e^{KS_R}(x-a)^{-1+K/k'(a)}/K \right\} + N\,|\,v_o(x)\,|/K \quad ,$$

that is to say,

$$|\,v_o(x)\,| \leqq k(x)\left\{1+J\,k(R)\,e^{KS_R}(x-a)^{-1+K/k'(a)}\right\}/(K-N) \quad .$$

If $K \geqq N$, it follows that $-1 + K/k'(a) \geqq 0$ and a constant C can be chosen such that $|v(x)| \leqq C\,k(x)$. If, however, $K < N$, then a constant can be found such that $|v_o(x)| \leqq C\,k(x)\ (x-a)^{-1+K/k'(a)}$.

If K is a negative number we may let $R = a$ and thus obtain from (19), since $\lim_{r=a} k(R)e^{K(s_R-z)} = 0$, the inequality,

$$|v_o(x)| \leqq -k(x)/K-e^{-Kz}\int_a^x k'(t)\,e^{Ks}dt/K \quad ,$$

or since $e^{Ks} \leqq e^{Kz}$,

$$|v_o(x)| \leqq -\left\{k(x) + \int_a^x k'(t)\ dt\right\}/K \leqq -2k(x)/K \quad .$$

These results may be summarized in the following lemma:

Lemma 2: If K is a negative number and R = a, then $|v_o(x)| \leqq -2k(x)/K$; if R is different from a and K is a positive number such that $K > |k'(x)|$, $K \geq |k'(a)|$, then $|v_o(x)| \leqq Ck(x)$, where C is a properly chosen constant; if K is a positive number such that $K < |k'(x)|$, and in particular, $K < k'(a)$, then $|v_o(x)| \leqq Ck(x)(x-a)^{-1+K/k'(a)}$.

Returning now to the equation,

$$u(x) = f(x) + \int_o^x [K\{1 + \Gamma(x, t)\}u(t)/k(t)]\,dt \quad ,$$

we express its solution formally by means of the series,

$$u(x) = u_o(x) + u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots\ldots \quad ,$$

where the functions are determined by means of the following algorithm:

$$u_o(x) = f(x) + \int_a^x \{K\, u_o(t)/k(t)\}\,dt \quad ,$$

$$u_i(x) = f_i(x) + \int_a^x \{K\, u_i(t)/k(t)\}\,dt \quad , \; i = 1, 2, 3, \ldots \, , \tag{20}$$

$$f_{i+1}(x) = \int_a^x K\{\Gamma(x,t)\, u(t)/k(t)\}\,dt \quad . \tag{21}$$

A majorant is obtained from the results of lemma 2. Since the solution of equation (20) is,

$$u_i(x) = e^{-K\int_x^b dt/k(t)} \int_R^x f_i'(t) e^{K\int_t^b dt/k(t)}\,dt \quad ,$$

a bound to $u_i(x)$ is given by,

$$|u_i(x)| \leqq |f_i'(x)|\,|v_o(x)| \quad .$$

Differentiating (21) we get,

$$f_{i+1}'(x) = K\,\Gamma(x,x)\,u_i(x)/k(x) + \int_a^x K\{\Gamma_x'(x, t)\, u_i(t)/k(t)\}\,dt \quad .$$

Since by the hypotheses of Theorem 4 we have $|\Gamma(x,x)| \leqq A(x-a)^\nu$, $|\Gamma_x'(x,t)| \leqq A$, a majorant for $f_{i+1}'(x)$ is obtained as follows:

$$|f_{i+1}'(x)| \leqq |K|A(x-a)^\nu |u_i(x)/k(x)| + |K|A\int_a^x |u_i(t)/k(t)|\,dt \quad ,$$

$$\leqq |K|A(x-a)^{\nu}|f_i'(x)|\ |v_o(x)/k(x)|+|K|A\int_a^x |v_o(t)/k(t)|\ |f_i'(t)|dt\ ,$$

$$\leqq |K|A(x-a)^{\nu}|f_i'(x)|\ |v_o(x)/k(x)|+|K|A|v_o(x)/k(x)|\ |f_i'(x)|(x-a)\ ,$$

or, provided $(x-a)<1$,

$$|f'_{i+1}(x)|\leqq 2|K|\ A(x-a)^{\nu}|f_i'(x)|\ |v_o(x)/k(x)|\ . \tag{22}$$

Substituting this in the inequality for $u_i(x)$ above we have,

$$|u_i(x)|\leqq 2|K|\ A(x-a)^{\nu}|f'_{i-1}(x)|\ |v_o(x)/k(x)|\ |v_o(x)|\ . \tag{23}$$

We now consider three cases:

Case 1. When $R=a$, $K<0$.

In this case, making use of lemma 2, we get $|v_o(x)|\leqq 2k(x)/|K|$, so that (22) and (23) become respectively,

$$|f'_{i+1}(x)|\leqq 4\ A(x-a)^{\nu}|f_i'(x)|\ ,$$

$$|u_i(x)|\leqq 8\ A(x-a)\ k(x)^{\nu}|f'_{i-1}(x)|/|K|\ .$$

Assuming that $|f_i'(x)|<N$, we get the inequality $|f'_{i+1}(x)|\leqq N\left\{4A(x-a)^{\nu}\right\}^{i+1}$, and hence obtain,

$$|u_i(x)|\leqq 2N\left\{4A(x-a)^{\nu}\right\}^{i}k(x)/|K| = C'M_1^i k(x)\ .$$

Case 2. When $R\neq a$, $K>|k'(x)|$, $K\geqq k(a)$.

By an argument similar to the one just given we get by means of lemma 2 the inequalities,

$$|f'_{i+1}(x)|\leqq N\left\{2KAC(x-a)^{\nu}\right\}^{i+1},$$

$$|u_i(x)|\leqq CNk(x)\left\{2KAC(x-a)^{\nu}\right\}^{i} = C'M_2^i k(x)\ .$$

Case 3. When $R\neq a$, $K<|k'(x)|$.

Again employing the results of lemma 2 we obtain,

$$|f'_{i+1}(x)|\leqq N\left\{2KAC(x-a)^{\nu-\mu}\right\}^{i+1},\ \text{where } 0<\mu<\nu\ ,$$

$$|u_i(x)|\leqq CN\left\{2KAC(x-a)^{\nu-\mu}\right\}^{i}k(x)\ (x-a)^{-\mu} = C'(x-a)^{-\mu}M_3^i k(x)\ .$$

Since the three majorant series, $\sum_{i=1}^{\infty} M_j^i$, $j=1, 2, 3$, converge for values of x such that $0\leqq (x-a)\leqq x'<1$, it follows that the series

$u(x) = \sum_{i=0}^{\infty} u_i(x)$ converges uniformly within the range (O,x'). In the first case u(x) is unique, but in case 2 and case 3 u(x) depends upon the arbitrary constant R and hence forms a one parameter family of solutions of the integral equation.

By an argument in no essential way different from that previously used, it may be proved that the function u(x) thus defined is actually a solution of the integral equation.

4. Application to the Integral Equation With an Infinite Limit. We have already considered in Theorem 1 a special case of the integral equation,

$$u(x) = f(x) + \int_x^{\infty} \left\{ G(x,t)\, u(t)/k(t)\, g(x) \right\} dt \quad , \tag{24}$$

where the assumption was made that $\lim_{x=\infty} \int_a^x dt/k(t)$ existed. An extension to the non-integrable case can now be made by means of an application of Theorem 4.

As we have previously shown there is no loss in generality if we assume that $g(x) \equiv 1$. If we make the transformation: $x = 1/y$, $t = 1/s$, equation (24) becomes,

$$u(1/y) = f(1/y) + \int_0^y \left\{ G(1/y, 1/s)\, u(1/s)/s^2 k(1/s) \right\} ds \quad . \tag{25}$$

The conditions of theorem 4 are now to be translated so that they apply to the present equation. We thus obtain the following theorem:

Theorem 5. The functions in equation (24), $g \equiv 1$, are assumed to satisfy the following conditions:

(1) In the interval $x \geqq a$, $f(x)$ and its first derivative are continuous, finite and such that $\lim_{x=\infty} f(x) = 0$.

(2) In the field $t \geqq x \geqq a$, $G(x,t)$ is continuous, finite and such that $\lim_{\substack{x=\infty \\ t=\infty}} G(x,t) = G$ exists and is different from zero.

(3) $\partial G(x,t)/\partial x$ and $\partial G(x,t)/\partial t$ are continuous and finite in the field $t \geqq x \geqq a$.

(4) $k(t)$ is positive, continuous and finite in the interval $x \geqq a$.

(5) There exists a finite value M such that for $t \geqq x \geqq a$ the following inequalities hold:

$k(x)/x < M$, $x|f'(x)| < M$, $x^2|\partial G(x,t)/\partial x| < M$, $t^2|\partial G(x,t)/\partial t| < M$.

Under these conditions there exists a unique solution of equation (24), $g(x) \equiv 1$, provided $G < 0$ and a one parameter family of solutions provided $G > 0$. These solutions remain finite and continuous, in the interval $x \geqq a$, with the possible exception of a finite number of points.

In proving this theorem let us consider the following identities:

$G(1/y, 1/s) \equiv G(x,t)$; $\partial G(1/y, 1/s)/\partial y \equiv -x^2 G'_x(x,t)$; $\partial G(1/y, 1/s)/\partial s \equiv -t^2 G'_t(x,t)$; $d\{s^2 k(1/s)\}/ds = -k'(t) + 2k(t)/t$; $d\{f(1/y)\}/dy = -x^2 f'(x)$; $s^2 k(1/s) = k(t)/t^2$; $f(1/s) = f(t)$.

The conditions stated in Theorem 5 follow from a translation of the conditions imposed by Theorem 4 upon the left-hand members of the identities into conditions to be satisfied by the right hand members.

Problems

1. Making the following transformation in equation (24) where $g(x) \equiv 1$: $x = e^y$, $t = e^s$, show that condition 5 of Theorem 5 may be replaced by the following assumption:

"that there exists a finite value M such that $k(x)/x < M$, $x(\log x)^2 |f'(x)| < M$, $x(\log x)^2 |G'_x(x,t)| < M$, $x(\log y)^2 | G'_t(x,t) | < M$." (Evans)

2. What do the conditions stated in the preceding problem become upon successive applications of the transformation?

5. Treatment of the Equation With an Infinite Limit by Means of Successive Substitutions. In previous sections we have discussed the equation,

$$u(x) = f(x) + \int_x^\infty K(x,t)\, u(t)\, dt\,, \tag{26}$$

by first reducing it thru transformations to an integral equation with a finite limit. We now study it directly by the method of successive substitutions developed in Section 1, of Part I.*

For this purpose we notice that the solution of equation (26) is given formally by the series,

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \quad , \tag{27}$$

*A special case of this equation has been treated by E. Hille in connection with a discussion of the oscillation properties of solutions of the Sturm-Liouville differential equation of second order for complex variables. The kernel used is $K(x,t) = \sin(t-x)\,\phi(t)$ where $\phi(t)$ satisfies the inequality $|\phi(t)| < M/|t|^{1+\nu}$ for in the region: $-\Theta \leqq \arg t \leqq \Theta$, $|t| \geqq \rho > 0$, where M and ν are positive constants. The function f(x) satisfies the equation $f''(x) + f(x) = 0$. See references 595 and 596. Also Trans. Amer. Math. Soc., vol. 23 (1923), pp. 350-385.

where we have $u_o(x) = f(x)$, $u_n(x) = \int_x^\infty K(x,t)u_{n-1}(t)dt$, $n = 1, 2, \ldots$.

We shall now prove the following theorem using the argument of Evans to whom this result is due:

Theorem 6. We shall assume that the functions in equation (26) satisfy the following conditions:

(1) $f(x)$ is continuous and finite in the interval $x \geqq a$;

(2) $K(x,t)$ is continuous and finite in the triangle $t \geqq x \geqq a$;

(3) $\int_x^\infty |K(x,t)|dt$ exists for $x \geqq a$;

(4) Values b and N exist such that $\int_x^\infty |K(x,t)|dt \leqq N < 1$, when $x \geqq b$.

*Under these conditions there exists a unique solution of (26) which is bounded in the interval $x \geqq a$. This solution is given by series (27).**

The proof depends upon the following theorem:†

Let $s_n(m)$ be a function of the two independent parameters n and m which satisfy the following conditions: (a) $\lim_{n=\infty} s_n(m) = s(m)$ exists; (b) $\lim_{m=\infty} s_n(m) = s_n$ exists; (c) for every positive value, there exists a number ν independent of m such that, $|s_n(m) - s_{n'}(m)| < \varepsilon$, for values of n and n′ greater than ν. We then conclude that $\lim_{m=\infty} s(m) = A$ exists, $\lim_{n=\infty} s_n = B$ exists and these limits are the same, $A = B$.

*The assumption of a bounded solution excludes the possibility of (26) having a one parameter family of solutions. For example, consider the equation:

$$u(x) = f(x) + A\int_x^\infty e^{B(x-t)}u(t)\,dt\ ,$$

whose formal solution is easily proved to be $u(x) = Ce^{(B-A)x} + f(x) + A\int_x^\infty e^{(B-A)(x-t)}f(t)\,dt$. If the integral is to exist in general then we must have $B - A > 0$, but in this case the function $Ce^{(B-A)x}$ is not bounded in the infinite interval. Hence we must set C equal to zero.

It should be noted also that the solution of the equation is not necessarily continuous. To secure this would require an auxiliary condition such as the continuity of the integral $\int_x^\infty |K(x,t)|dt$. An example given by Evans (See reference 436, vol. 20 (2), p. 8) follows:

Let $f(x) = A$ and $K(x,t) = \begin{cases} \left\{t - \psi(x)\right\}e^{-\left\{t-\psi(x)\right\}}, & t \geqq \psi(x),\ x < x_0; \\ 0 \quad \text{(in the remainder of the field)}, \end{cases}$ where $t = \psi(x)$ is a continuous function for $0 \leqq x < x_0$ such that $\psi(x) > x_0$ when $x < x_0$ and $\psi(x) = \infty$ when $x \to x_0$.

Then $\int_x^\infty |K(x,t)|dt = \begin{cases} 1/2 \text{ for } x < x_0 \\ 0 \text{ for } x \geqq x_0 \end{cases}$, and the solution of the equation with $K(x,t)$ for kernel is $u(x_0 + 0) = A$, $u(x_0 - 0) = 2A$.

†Osgood: Funktionentheorie, vol. I, (1907) p. 519; second edition (1912), p. 593.

In order to apply this theorem to our problem we consider the equation,

$$u(x) = f(x) + \int_x^m K(x,t)\, u(t)\, dt \quad .$$

Making the abbreviations,

$$u_o(x) = f(x),\ u_n(x,m) = \int_x^m K(x,t) u_{n-1}(x,m)\, dt,\ n = 1, 2, \ldots .$$

we have from theorem 1 of part I for finite values of m the unique solution,

$$u(x,m) = \sum_{n=o}^{\infty} u_n(x,m) \quad . \tag{28}$$

The question at issue is whether or not we can carry over the results of that theorem to the case where $m \to \infty$. In order to answer this we consider the sum,

$$U_n(x,m) = \sum_{n=o}^{n} u_n(x,m) \quad ,$$

and apply to it the criteria of the theorem on double limits. We see at once from (28) that condition (a) is met since

$$\lim_{n=\infty} U_n(x,m) = u(x,m)$$

exists.

We next prove that if M, m, and m_o are three numbers such that $M > m > m_o$ and A and B are numbers such that $A \geqq B \geqq b$, then for any preassigned positive value ε a value of m_o can be found such that the following inequalities hold:

$$|u_n(x,M) - u_n(x,m)| \leqq \varepsilon \quad ,$$

$$\int_B^A |u_n(t,M) - u_n(t,m)|\, dt \leqq \varepsilon \quad .$$

The proof is made by induction. The inequalities are seen to hold for n = 0, since $u_o(x) = f(x)$. Assuming that they hold for n, it is then merely necessary to prove that they hold for n + 1. We thus have for the first,

$$|u_{n+1}(x,M) - u_{n+1}(x,m)| \leqq \left|\int_m^M K(x,t)\, u_n(t,M) dt + \right.$$

$$\left. \int_{m_o}^m K(x,t)\left\{u_n(t,M) - u_n(t,m)\right\} dt + \int_x^{m_o} K(x,t)\left\{u_n(t,M) - u_n(t,m)\right\} dt\right| .$$

The upper bound of $|u_n(t,M)|$ is by hypothesis $N^{n-1}F$, where F is the majorant of f(x); hence if we designate by K the majorant of K(x,t), the right-hand member of the inequality reduces to,

$$3N^{n-1}F\int_m^M |K(x,t)|dt + K\int_x^{m_o} |u_n(t,M) - u_n(t,m)|dt \quad .$$

Since by hypothesis both of the inner integrals are arbitrarily small for a properly chosen value of m, we may write,

$$\int_m^M |K(x,t)|dt \leqq \varepsilon/6N^{n-1}F \quad ,$$

$$\int_x^{m_o} |u_n(t,M) - u_n(t,m)|dt \leqq \varepsilon/2K \quad .$$

From these inequalities, combined with the preceding, we get,

$$|u_{n+1}(x,M) - u_{n+1}(x,m)| < \varepsilon \quad ,$$

which was to be proved.

The second inequality, $\int_B^A |u_n(t,M) - u_n(t,m)|dt < \varepsilon$, is derived without any essential change in the argument.

Making use of these conclusions we see that $\lim_{m=\infty} u_n(x,m)$ exists and that $\lim_{\substack{m=\infty \\ M'>m}} \int_B^A |u_n(t,M) - u_n(t,m)|dt$ converges uniformly to zero.

Hence, since $U_n(x,m)$ is a finite sum of the functions $u_n(x,m)$, we are able to conclude that

$$\lim_{m=\infty} U_n(x,m)$$

exists.

The third part of the limit theorem, which postulates the uniform convergence of $U_n(x,m)$ is immediately established from the inequality

$$|U_n(x,m)| \leqq F(1+N+N^2+\ldots+N^n) \leqq F/(1-N), \quad x > b \quad .$$

Combining these results we reach the desired conclusion that $\lim_{m=\infty} \left\{ \lim_{n=\infty} U_n(x,m) \right\} = u(x)$ exists and is equal to the limit,

$$\lim_{n=\infty} \left\{ \lim_{m=\infty} U_n(x,m) \right\} \quad .$$

In order to prove that u(x) furnishes a solution of the integral equation we first observe that,

$$\int_x^m K(x,t)\, u(t,m)dt = u(x,m) - f(x) \quad ,$$

and from the conclusions reached above, that $\lim_{m=\infty} \int_x^m K(x,t)\, u(t,m)dt$ exists and is equal to u(x)—f(x).

But this limit is identical with the following:

$$\lim_{m=\infty} \int_x^\infty K(x,t)\, u(t,m)\, dt \, ,$$

since we have,

$$\left| \int_x^\infty K(x,t)\, u(t,m)\, dt - \int_x^m K(x,t)\, u(t,m)\, dt \right| \leqq$$

$$\int_m^\infty |K(x,t)||u(t,m)|dt \leqq \left\{K/(1-N)\right\} \int_m^\infty |K(x,t)|dt \leqq Ke/(1-N) \quad .$$

Hence we have proved that

$$\lim_{m=\infty} \left\{ \lim_{M=\infty} \int_x^M K(x,t)\, u(t,m)\, dt \right\} \quad , \tag{29}$$

exists.

Introducing the notation of the resolvent kernel we see that we can write u(x,m) in the form,

$$u(x,m) = f(x) + \int_m^x k(x,t)\, f(t)\, dt \quad ,$$

where we use the customary abbreviation, $k(x,t) = -K(x,t) + K_2(x,t) - K_3(x,t) + \ldots\ldots$.

From the conditions imposed by hypothesis upon K(x,t) the following inequalities are derived:

$$|k(x,t)| \leqq K/(1-N); \int_x^\infty |k(x,t)|\, dt \leqq N/(1-N) \quad .$$

Consequently the function $f(x,m) = \int_x^m k(x,t)\, f(t)\, dt$ has the properties: (a) that $\lim_{m=\infty} f(x,m)$ exists and (b) that $\lim_{m=\infty} \int_A^B |f(t,M) - f(t,m)|dt = 0$, $M > m$. Therefore these properties are possessed also by U(x,m) and we have in consequence the inequality,

$$\int_A^B |K(t,s)||u(s,M) - u(s,m)|ds\ dt \leqq K\int_A^B |u(s,M) - u(s,m)|ds \leqq \epsilon$$

for M and m sufficiently large.

Our conclusion from this is that,

$$\lim_{m=\infty} \int_A^B K(x,t)\ u(t,m)\ dt \quad ,$$

exists for $x \geqq a$, $B \geqq A \geqq a$.

Thus we have established the existence of the two limits,

$$\lim_{m=\infty} \int_x^M K(x,t)\ u(t,m)\ dt \quad ,$$

$$\lim_{M=\infty} \int_x^M K(x,t)\ u(t,m)\ dt \quad ,$$

and the uniformity of the latter with respect to m. Hence the limit theorem applies; we are able to invert the order of the limits in (29) and assert that,

$$\lim_{m=\infty} \left\{ \lim_{M=\infty} \int_x^M K(x,t)\ u(t,m)\ dt \right\} = \lim_{M=\infty} \left\{ \lim_{m=\infty} \int_x^M K(x,t)\ u(t,m)\ dt \right\} = \int_x^\infty K(x,t)\ dt \quad .$$

This conclusion proves that the function u(x), whose existence was previously established, actually furnishes a solution of the equation.

The question of the uniqueness of this solution depends upon the assumption that u(x) is to be finite and integrable in the interval $x \geqq b$. If there existed a second solution of the equation then the difference between the two solutions would satisfy the homogeneous equation,

$$v(x) = \int_x^\infty K(x,t)\ v(t)\ dt \quad .$$

But since the right-hand member of this equation is finite and integrable we have,

$$v(x) = \int_x^\infty K(x,t)\ dt \int_t^\infty K(t,s)\ v(s)\ ds \quad ,$$

where $|v(x)| \leqq CN^2$. Continuing the substitution of v(s) in the second member of the equation, we thus readily prove that $|v(x)| \leqq CN^{n+1}$ and v(x) thus reduces to zero in the interval $x \geqq b$.

The proof that the solution is also unique in the interval $a \leqq x \leqq b$ is easily established by writing the equation in the form

$$u(x) = \psi(x) + \int_x^b K(x,t)\, u(t)\, dt \quad ,$$

where

$$\psi(x) = f(x) + \lim_{m=\infty} \int_b^m K(x,t)\, u(t)\, dt \quad . \tag{30}$$

The function $\psi(x)$ obtained by substituting the unique value of $u(t)$, $t \geqq b$, in the second equation exists and is unique. Hence the solution of the first equation is also unique in the interval $a \leqq x \leqq b$ by virtue of Theorem 1, Part I.

6. Asymptotic Solution of the Equation with an Infinite Limit. If in the theorem of the preceding section we remove the restriction that the solution shall be bounded at infinity a further extension is possible. The following theorem is due to C. E. Love:*

Theorem 7. If the integral $\int_x^\infty |K(x,t)\, f(t)|\, dt$ *converges for* $x \geqq a$, *converges uniformly for* x *in the interval* $a \leqq x \leqq b$, *and is less than* $N|f(x)|$, $0 < N < 1$ *for* $x \geqq a$, *where* N *is a constant independent of* x, *then a continuous solution of equation (26) exists which is expressible in the form* $u(x) = f(x)\rho(x)$, *where* $\rho(x)$ *is a continuous function such that* $|\rho(x)| < 1/(1-N)$.

Writing the formal solution in the form,

$$u(x) = \sum_{n=o}^{\infty} u_n(x) \quad , \tag{31}$$

where $u_o(x) = f(x)$, $u_n(x) = \int_x^\infty K(x,t) u_{n-1}(t) dt$, we first establish the absolute convergence of the series for $x \geqq a$. To do this let us assume the inequality,

$$|u_{n-1}(x)| \leqq N^{n-1}|f(x)|, \quad n = 1, 2, \ldots \ldots \quad ,$$

and show that the inequality then holds for the subscript n.

Since the integral $\int_x^\infty |K(x,t)\, f(t)|\, dt$ by hypothesis converges uniformly in the range $a \leqq x \leqq b$, we can find for every positive value ε, two numbers m and M such that $\int_m^M |K(x,t)\, f(t)|\, dt < \varepsilon$, where x lies in the restricted range.

*Reference 445.

Hence we have,

$$\left|\int_m^M K(x,t)\, u_{n-1}(t)dt\right| \leqq \int_m^M |K(x,t)u_{n-1}(t)|dt \leqq N^{n-1}\int_m^M |K(x,t)\, f(t)|\, dt < \epsilon.$$

The continuity and uniform convergence of $u_n(x)$ in the restricted range is established by these inequalities. The continuity at any point in the open range $x \geqq a$ then follows since $x = b$ can always be chosen sufficiently large so as to include any preassigned point.

Moreover, from our assumption, it is easily seen that

$$|u_n(x)| \leqq \int_x^\infty |K(x,t)\, u_{n-1}(t)|dt \leqq N^{n-1}\int_x^\infty |K(x,t)\, f(t)|dt < N^n|f(x)| \quad .$$

Hence from the fact that we have $|u_0(x)| \leqq f(x)|$, that is to say when $n = 0$, the desired induction is immediately obtained.

The absolute convergence of series (31) follows from this since we have,

$$\sum_{n=0}^{\infty} |u_n(x)| < |f(x)|/(1-N) \quad . \tag{32}$$

Hence by the Weierstrass test $u(x)$, for x in the restricted range, is the sum of a uniformly convergent series of continuous functions and is, therefore, itself a continuous function.

That $u(x)$ is actually a solution of the equation can be proved by means of an argument similar to the one given in the preceding section. Since no essential change is necessary in this proof, the discussion will be omitted.

The uniqueness of the solution, however, does not follow in this case and more than one continuous solution of the equation is possible. The essential difference between Theorem 6 and Theorem 7 is found in the hypothesis regarding the finiteness of $f(x)$ and of the resulting solution. Only by imposing an upper bound upon these two functions in the infinite interval is it possible to affirm the uniqueness of the solution.

An example is furnished by the equation,

$$u(x) = f(x) + A\int_x^\infty e^{B(x-t)}u(t)\, dt \quad .$$

whose formal solution is,

$$u(x) = Ce^{(B-A)x} + f(x) + A\int_x^\infty e^{(B-A)(x-t)}f(t)\, dt \quad .$$

If we assume $f(x) = e^{kx}$, $A-B+k<0$, this solution reduces to,

$$u(x) = Ce^{(B-A)x} + (B-k)e^{kx}/(B-A-k) \quad .$$

Making the further assumptions that $|A/(B-k)| < 1$, $k-B < 0$, this function is seen to furnish a solution of the equation under the hypotheses of the theorem. If in addition B—A and k are positive the solution is clearly not finite at infinity nor is it unique. Values of the letters which satisfy the above conditions are: $A = 1/2$, $B = 2$, $k = 1$.

A generalization of theorem 7 may be stated as follows:

Theorem 8. If there exists a continuous, positive, real function $h(x)$ such that the integrals,

$$I = \int_x^\infty |K(x,t)\, f(t)|dt,\ J = \int_x^\infty |K(x;t)\ |\ h(t)\ dt\ ,$$

converge in the interval $x \geqq a$, converge uniformly in the closed interval $a \leqq x \leqq b$, and satisfy in the open interval the inequalities $I < h(x)$, $J < Nh(x)$, $(0 < N < 1)$, where N is a constant independent of x, then equation (26) has in (a,∞) a continuous solution $u(x)$ expressible in the form,

$$u(x) = f(x) + h(x)\, \rho(x),$$

where $\rho(x)$ is a continuous function subject to the inequality

$$|\ \rho(x)\ | < 1/\ (1-N)\ .$$

The proof of this theorem is essentially equivalent to that of the preceding theorem and need not be given in detail. By obvious modifications we can establish the identities,

$$\sum_{n=1}^{\infty} |u_n(x)| < h(x)/(1-N)\ ,$$
$$|u(x) - f(x)| < h(x)/(1-N)\ .$$

The theorem is essentially a statement of these results.

Making use of the conclusions of Theorem 7 it is possible to establish the following generalization:

Theorem 9. If there exists a continuous, positive, real function $g(x)$ such that the integral $\int_x^\infty g(t)\, dt$ exists and such that in the triangle $t \geqq x \geqq a$ the inequality $|\ K(x,t)\, f(t)\ | < |\ f(x)\ |\ g(t)$ holds, then equation (26) has a continuous solution asymptotic to $f(x)$,

$$u(x) \sim f(x).$$

This theorem follows from Theorem 7 if we choose N to be any number satisfying the inequalities:

$$\int_x^\infty g(t)\ dt < N\ ,\quad 0 < N < 1\ .$$

It is obvious that if x is taken sufficiently large, N may be chosen as small as we wish and in particular some number between 0 and 1. That this implies no loss of generality in the final result will be apparent when we consider the device employed by Evans to extend a solution known for an interval (ab) into the interval beyond b. (See end of section 1, part I).

Making use of the above inequality we then obtain,

$$\int_x^\infty |K(x,t)\, f(t)|\, dt < |f(x)| \int_x^\infty g(t)\, dt < N|f(x)| \quad .$$

The uniform convergence of the integral in the closed interval (ab) follows from the existence of the integral $\int_x^\infty g(t)dt$ and the inequality: $|K(x,t)\, f(t)| < F\, g(t)$, where F is the upper bound of f(x) in (ab).*

These facts meet the conditions imposed by Theorem 7 and the existence of a continuous solution subject to the inequality,

$$u(x) = f(x)\ \rho(x),\ |\rho(x)| \leqq (1 + N + N^2 + \ldots\ldots) \quad ,$$

is thus established. Since for sufficiently large values of x, N can be made as small as we please, the solution is obviously asymptotic to f(x),

$$u(x) \sim f(x) \quad .$$

Another generalization of similar kind is contained in the following theorem:

Theorem 10. If $g(x)$ and $h(x)$ are continuous, positive, real functions such that the following conditions are satisfied:

(1) $\int_x^\infty g(t)\, dt$ *exists;* *(2)* $|K(x,t)\, f(t)| < h(x)\, g(t)$, *for* $t \geqq x \geqq a$; *(3)* $|K(x,t)|\, h(t) < h(x)\, g(t)$ *for* $t \geqq x \geqq a$, *then a continuous solution of equation (26) exists in the form,*

$$u(x) = f(x) + h(x)\, \varepsilon\, (x), \quad \lim_{x=\infty} \varepsilon\, (x) = 0.$$

The proof is effected by reducing the problem to the circumstances of theorem 8. It is obvious that conditions (2) and (3) assure us that the integrals: $\int_x^\infty |K(x,t)\, f(t)|dt$, $\int_x^\infty |K(x,t)|\, h(t)\, dt$, possess

*See T. J. Bromwich, Theory of Infinite Series, 2d ed. (1926), p. 481; 1st ed. (1908), p. 434.

the properties postulated by theorem 8. Hence a continuous solution exists of the form,

$$u(x) = f(x) + h(x)\ \rho\ (x) \quad ,$$

where $h(x)\ \rho\ (x) = \sum_{n=1}^{\infty} u_n(x)$.

But from condition (2), for sufficiently large values of x, we obtain the inequality,

$$|u_1(x)| < h(x)\int_x^{\infty} g(t)\ dt < h(x)\ N\ , \quad 0 < N < 1.$$

Moreover we have $\sum_{n=2}^{\infty} |\ u_n(x)| < h(x)\ N/(1—N)$, by means of an argument identical with that employed in the proof of theorem 7. Hence we arrive at the inequality,

$$h(x)\ \rho\ (x) < h(x)\left\{N + N/(1—N)\right\} \quad ,$$

and since N can be made arbitrarily small, if x is chosen sufficiently large, the conclusion of the theorem is reached.

Problem

If in equation (26) the kernel and known function are of the form:

$$K(x,t) \sim e^{p(x)}x^b e^{—r(t)}t^c(K_{00}+K_{10}/x + K_{01}/t + K_{20}/x^2 + \ldots .) \quad ,$$
$$f(x) \sim e^{q(x)}x^a\ (f_0+ f_1/x + f_2/x^2 + \ldots .) \quad ,$$

where $p(x)$, $q(x)$, and $—r(x)$ are polynomials of positive sign for sufficiently large values of x and such that $R[p(x)] < R[q(x)] < R[r(x)]$ in which $R[\ ]$ means the "real part of", prove that a continuous solution exists asymptotic to $f(x)$, $u(x) \sim f(x)$. (Love)

Hint: Letting $g(x) = C|e^{q(x)—r(x)}|x^k$, where C and k are suitably chosen constants, and $h(x) = |e^{p(x)}x^b|$, bring the existence proof under the conditions of Theorem 10.

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